ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \int_{\frac{\pi}{4}}^{\pi} \left(\frac{\ln(\sqrt{x})}{x} - \frac{ln^3(y+1)}{y^3}\right) dxdy$$

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$$I = \underbrace{\int_{\frac{\pi}{4}}^{\pi} \frac{\ln(\sqrt{x})}{x} dx}_{I_1} - \frac{3\pi}{4} \underbrace{\int_0^1 \frac{ln^3(y+1)}{y^3} dy}_{I_2}$$

$$I_1 = \int_{\frac{\pi}{4}}^{\pi} \frac{\ln(\sqrt{x})}{x} dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\pi} \frac{\ln(x)}{x} dx \overset{\ln(x)\to x}{\cong} \frac{1}{2}\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln(\pi)} x dx = \frac{x^2}{4}\Bigg|_{\ln\left(\frac{\pi}{4}\right)}^{\ln(\pi)} =$$

$$= \frac{ln^2(\pi) - ln^2\left(\frac{\pi}{4}\right)}{4} = \ln(2)\ln\left(\frac{\pi}{2}\right)$$

$$I_2 = \int_0^1 \frac{ln^3(y+1)}{y^3} dy \overset{\frac{1}{1+y}=y}{\cong} -\int_{\frac{1}{2}}^1 \frac{yln^3(y)}{(1-y)^3} dy = -\underbrace{\int_0^1 \frac{yln^3(y)}{(1-y)^3} dy}_{I_3} + \underbrace{\int_0^{\frac{1}{2}} \frac{yln^3(y)}{(1-y)^3} dy}_{I_4}$$

$$I_3 = \int_0^1 \frac{yln^3(y)}{(1-y)^3} dy = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)\int_0^1 y^{n-1} ln^3(y) dy = -3\sum_{n=2}^{\infty} \frac{n(n-1)}{n^4} =$$

$$= 3\sum_{n=1}^{\infty}\left(\frac{1}{n^3} - \frac{1}{n^2}\right) = 3\zeta(3) - \frac{\pi^2}{2}$$

$$I_4 = \int_0^{\frac{1}{2}} \frac{yln^3(y)}{(1-y)^3} dy \overset{y\to\frac{y}{2}}{\cong} 2\int_0^1 \frac{yln^3\left(\frac{y}{2}\right)}{(2-y)^3} dy$$

$$= \sum_{n=3}^{\infty} \frac{(n-1)(n-2)}{2^n} \int_0^1 y^{n-2}(\ln(y) - \ln(2))^3 dy =$$

$$\sum_{n=3}^{\infty} \frac{(n-1)(n-2)}{2^n} \left(\int_0^1 y^{n-2} ln^3(y) dy - 3\ln(2)\int_0^1 y^{n-2} ln^2(y) dy + 3\, ln^2(y)\int_0^1 y^{n-2} \ln(y)\, dy - \right.$$

$$-\ln^3(2)\int_0^1 y^{n-2}dy) = \sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{2^n}\left(-\frac{3!}{(n-1)^4}-\frac{3\ln(2)\,2!}{(n-1)^3}-\frac{3\ln^2(2)}{(n-1)^2}-\frac{\ln^3(2)}{n-1}\right) =$$

$$-6\sum_{n=3}^{\infty}\frac{n-2}{2^n(n-1)^3}-6\ln(2)\sum_{n=3}^{\infty}\frac{n-2}{2^n(n-1)^2}-3\ln^2(2)\sum_{n=3}^{\infty}\frac{n-2}{2^n(n-1)}-\ln^3(2)\sum_{n=3}^{\infty}\frac{n-2}{2^n} =$$

$$= 3\sum_{n=2}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{n^2}-\frac{\left(\frac{1}{2}\right)^n}{n^3}\right)-3\ln(2)\sum_{n=2}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{n}-\frac{\left(\frac{1}{2}\right)^n}{n^2}\right)-\frac{3\ln^2(2)}{2}\sum_{n=2}^{\infty}\left(\frac{1}{2^n}-\frac{\left(\frac{1}{2}\right)^n}{n}\right)-\frac{\ln^3(2)}{4}\sum_{n=1}^{\infty}\frac{n}{2^n} =$$

$$3\left[Li_3\left(\frac{1}{2}\right)-Li_2\left(\frac{1}{2}\right)\right]-3\ln(2)\left[Li_1\left(\frac{1}{2}\right)-Li_2\left(\frac{1}{2}\right)\right]-\frac{3\ln^2(2)}{2}\left[1-Li_1\left(\frac{1}{2}\right)\right]-\frac{\ln^3(2)}{4}Li_{-1}\left(\frac{1}{2}\right) =$$

$$3\left(\frac{\ln^2(2)}{6}-\frac{\pi^2\ln(2)}{12}+\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}+\frac{\ln^2(2)}{2}\right)-3\ln(2)\left(\ln(2)-\frac{\pi^2}{12}+\frac{\ln^2(2)}{2}\right)-\frac{3\ln^2(2)}{2}+$$

$$+\frac{3\ln^3(2)}{2}-\frac{\ln^3(2)}{2} = \frac{21}{8}\zeta(3)-\frac{\pi^2}{4}-3\ln^2(2)$$

$$I_1 = \ln(2)\ln\left(\frac{\pi}{2}\right)$$

$$I_2 = \int_0^1\frac{\ln^3(y+1)}{y^3}dy = -\underbrace{\int_0^1\frac{y\ln^3(y)}{(1-y)^3}dy}_{I_3}+\underbrace{\int_0^{\frac{1}{2}}\frac{y\ln^3(y)}{(1-y)^3}dy}_{I_4} = -\left(3\zeta(3)-\frac{\pi^2}{2}\right)+$$

$$+\frac{21}{8}\zeta(3)-\frac{\pi^2}{4}-3\ln^2(2) = -\frac{3}{8}\zeta(3)+\frac{\pi^2}{4}-3\ln^2(2)$$

$$I = \underbrace{\int_{\frac{\pi}{4}}^{\pi}\frac{\ln(\sqrt{x})}{x}dx}_{I_1}-\frac{3\pi}{4}\underbrace{\int_0^1\frac{\ln^3(y+1)}{y^3}dy}_{I_2} = \ln(2)\ln\left(\frac{\pi}{2}\right)-\frac{3\pi}{4}\left(-\frac{3}{8}\zeta(3)+\frac{\pi^2}{4}-3\ln^2(2)\right)$$

$$= \frac{9}{32}\pi\left(\zeta(3)+8\ln^2(2)\right)+\ln(2)\ln\left(\frac{\pi}{2}\right)-\frac{3\pi^3}{16}$$