ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \frac{x\ln(x)}{(1+x)(1+x^2)}dx$$

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$$\int_0^1 \frac{x\ln(x)}{(1+x)(1+x^2)}dx \overset{\text{Partial fraction}}{\triangleq} \frac{1}{2}\int_0^1 \frac{\ln(x)}{1+x^2}dx + \underbrace{\frac{1}{2}\int_0^1 \frac{x\ln(x)}{1+x^2}dx}_{x^2\to x} -$$

$$\frac{1}{2}\int_0^1 \frac{\ln(x)}{1+x}dx \overset{\text{Note section}}{\triangleq} -\frac{G}{2} - \frac{3}{8}\int_0^1 \frac{\ln(x)}{1+x}dx = -\frac{G}{2} + \frac{3}{8}\eta(2) =$$

$$= -\frac{G}{2} + \frac{3}{16}\zeta(2) = \frac{\pi^2}{32} - \frac{G}{2}$$

$$\int_0^1 \frac{x\ln(x)}{(1+x)(1+x^2)}dx = \frac{\pi^2}{32} - \frac{G}{2}$$

***Note section*:**

1) $\int_0^1 \frac{\ln\left(\frac{1}{t}\right)}{1+t^2}dt = G$ $(Catalan's\ constant)$

2) $\eta(z) = \frac{(-1)^{z-1}}{\Gamma(z)}\int_0^1 \frac{ln^{z-1}(x)}{1+x}dx,\ \ \Re(z) > 0$

3) $\eta(s) = (1-2^{1-s}).\zeta(s)$