ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \int_0^1 \frac{x^2\ln(x) + ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\int_0^1 \frac{x^2\ln(x) + ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx = \int_0^1 \frac{x^2\ln(x)}{1+x+x^2+x^3}dx + \frac{1}{4}\int_0^1 \frac{ln^2(\sqrt{x})}{1+x+x^2+x^3}dx =$$

$$\int_0^1 \frac{(1-x)x^2\ln(x)}{(1-x)(1+x+x^2+x^3)}dx + \frac{1}{4}\int_0^1 \frac{(1-x)ln^2(\sqrt{x})}{(1-x)(1+x+x^2+x^3)}dx =$$

$$\int_0^1 \frac{(1-x)x^2\ln(x)}{1-x^4}dx + \frac{1}{4}\int_0^1 \frac{(1-x)ln^2(\sqrt{x})}{1-x^4}dx = \frac{1}{16}(\int_0^1 \frac{x^{\frac{3}{4}-1}\ln(x)}{1-x}dx - \int_0^1 \frac{\ln(x)}{1-x}dx) +$$

$$\frac{1}{256}(\int_0^1 \frac{x^{\frac{1}{4}-1}ln^2(x)}{1-x}dx - \int_0^1 \frac{x^{\frac{1}{2}-1}ln^2(x)}{1-x}dx)$$

$$\left\{\therefore \int_0^1 \frac{z^{n-1}ln^k(z)}{1-z}dz = -\psi^{(k)}(n), \forall\, k \in N\right\}$$

$$= \frac{1}{16}\left(-\psi^{(1)}\left(\frac{3}{4}\right) + \psi^{(1)}(1)\right) + \frac{1}{256}\left(-\psi^{(2)}\left(\frac{1}{4}\right) + \psi^{(2)}\left(\frac{1}{2}\right)\right)$$

$$\{\therefore \psi^{(k)}(n) = (-1)^{k+1}\Gamma(k+1)\,\zeta(k+1,n)\}$$

$$= \frac{1}{16}\left(-(\pi^2 - 8G) + \frac{\pi^2}{6}\right) + \frac{1}{256}\left(-\left(-2\pi^3 - 56\zeta(3)\right) - 14\zeta(3)\right)$$

$$= \frac{1}{16}\left(8G - \frac{5\pi^2}{6}\right) + \frac{1}{256}(42\zeta(3) + 2\pi^3)$$

$$\int_0^1 \frac{x^2\ln(x) + ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx = \frac{G}{2} + \frac{21\zeta(3)}{128} + \frac{\pi^3}{128} - \frac{5\pi^2}{96}$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\Omega = \int_0^1 \frac{x^2\ln(x) + ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx = \int_0^1 \frac{x^2\ln(x)}{(x+1)(x^2+1)}dx + \int_0^1 \frac{\ln(\sqrt{x})}{(x+1)(x^2+1)}dx = \Omega_1 + \Omega_2$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\Omega_1=\int_0^1\frac{x^2\ln(x)}{(x+1)(x^2+1)}dx=\frac{1}{2}\int_0^1\frac{x\ln(x)}{x^2+1}dx-\frac{1}{2}\int_0^1\frac{\ln(x)}{x^2+1}dx+\frac{1}{2}\int_0^1\frac{\ln(x)}{x+1}dx=$$

$$\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln(x)\,dx-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(x)\,dx+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n}\ln(x)\,dx=$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=\frac{G}{2}-\frac{5}{8}\eta(2)=\frac{G}{2}-\frac{5\pi^2}{96}$$

$$\Omega_2=\int_0^1\frac{\ln(\sqrt{x})}{(x+1)(x^2+1)}dx=\frac{1}{4}\int_0^1\frac{\ln(x)}{(x+1)(x^2+1)}dx=\frac{1}{8}\int_0^1\frac{ln^2(x)}{x^2+1}dx-\frac{1}{8}\int_0^1\frac{xln^2(x)}{x^2+1}dx+$$

$$\frac{1}{8}\int_0^1\frac{ln^2(x)}{x+1}dx=\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}ln^2(x)dx-\frac{1}{8}\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln(x)\,dx+$$

$$\frac{1}{8}\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n}ln^2(x)dx=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\frac{1}{32}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}=$$

$$=\frac{1}{4}\cdot\frac{\pi^3}{32}+\frac{7}{32}\cdot\frac{3}{4}\zeta(3)=\frac{\pi^3}{128}+\frac{21}{128}\zeta(3)$$

$$\Omega=\Omega_1+\Omega_2=\frac{G}{2}+\frac{\pi^3}{128}+\frac{21}{128}\zeta(3)-\frac{5\pi^2}{96}$$