ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1\int_1^\infty \frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1\int_1^\infty \frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy=\int_0^1\frac{Li_3(-y)}{(y+1)^3}dy\,.\int_1^\infty\frac{ln^3(x)}{(x+1)^2}dx=A\,.B$$

$$A=\int_0^1\frac{Li_3(-y)}{(y+1)^3}dy\ \left\{u=Li_3(-y),\quad \frac{du}{dy}=\frac{Li_2(-y)}{y},\quad v=-\frac{1}{2(y+1)^2},\quad dv=\frac{1}{(y+1)^3}\right\}$$

$$A=\frac{-Li_3(-y)}{2(y+1)^2}\Big|_0^1+\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y(y+1)^2}dy=\frac{3\zeta(3)}{32}+\frac{1}{32}+\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y(y+1)^2}dy$$

$$\begin{cases} u=Li_2(-y)\,,\ \dfrac{du}{dy}=\dfrac{-\ln(y+1)}{y}\\ dv=\dfrac{1}{y(y+1)^2},\ \ v=\dfrac{1}{y+1}+\ln(y)-\ln(y+1)\end{cases}$$

$$A=\frac{3\zeta(3)}{32}+\Bigg[\ -\frac{Li_2(-y)}{y+1}\Big|_0^1+Li_2(-y)\ln(y)\Big|_0^1-Li_2(-y)\ln(y+1)\Big|_0^1$$
$$+\int_0^1\frac{\ln(y+1)}{y(y+1)}dy+\int_0^1\frac{\ln(y)\ln(y+1)}{y}dy-\int_0^1\frac{ln^2(y+1)}{y}dy\ \Bigg]$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}\Big(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)$$
$$+\int_0^1\frac{\ln(y+1)}{y}dy$$
$$-\int_0^1\frac{\ln(y+1)}{y+1}dy+\int_0^1\frac{\ln(y)\ln(y+1)}{y}dy-\int_0^1\frac{ln^2(y+1)}{y}dy\Big)$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}\Big(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)$$
$$-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 y^{n-1}dy-\frac{1}{2}ln^2(y+1)\Big|_0^1-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 y^{n-1}ln^2(y)dy$$
$$-\int_0^2\frac{ln^2(y)}{y}dy\Big)\ =$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\frac{1}{2}ln^2(2)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}+\sum_{n=1}^{\infty}\int_1^2 y^{n-1}ln^2(y)dy$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)+\frac{\zeta(2)}{2}-\frac{1}{2}ln^2(2)-\frac{3}{4}\zeta(3)-\frac{\zeta(3)}{4})$$

$$A=\frac{3\zeta(3)}{32}+\frac{\zeta(2)}{8}+\frac{\zeta(2)}{2}\ln(2)-\frac{1}{4}ln^2(2)-\frac{\zeta(3)}{2}$$

$$B=\int_1^{\infty}\frac{ln^3(x)}{(x+1)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_1^0\frac{ln^3\left(\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)^2}-\frac{dx}{x^2}=-\int_0^1\frac{x^2ln^3(x)}{(x+1)^2}\frac{dx}{x^2}=-\int_0^1\frac{ln^3(x)}{(x+1)^2}dx$$

$$B=3\int_0^1\frac{ln^2(x)}{1+x}dx=-3\sum_{n=1}^{\infty}\int_0^1 x^{n-1}ln^2(x)dx=-3\sum_{n=1}^{\infty}(1)^n\left(\frac{2}{n^3}\right)=-6\sum_{n=1}^{\infty}\frac{1}{n^3}=-6.\left(-\frac{3}{4}\zeta(3)\right)$$
$$=\frac{9}{2}\zeta(3)$$

$$I=A\ .\ B$$

$$I=\left[-\frac{13}{32}\zeta(3)+\frac{\zeta(2)}{8}+\frac{\zeta(2)}{4}\ln(2)-\frac{1}{4}ln^2(2)\right].\ \frac{9}{2}\zeta(3)$$

$$I=-\frac{117}{64}\zeta^3(3)+\frac{9}{16}\zeta(3)\zeta(2)+\frac{9}{8}\zeta(3)\zeta(2)\ln(2)-\frac{9}{8}\zeta(3)ln^2(2)$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1\int_1^{\infty}\frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy$$

$$H=\left((\int_0^1\frac{Li_3(-y)}{(y+1)^3}dy\ ).(\int_1^{\infty}\frac{ln^3(x)}{(x+1)^2}dx)\right)$$

$$I=\int_1^{\infty}\frac{ln^3(x)}{(x+1)^2}dx\overset{x\to\frac{1}{x}}{\cong}-\int_0^1\frac{ln^3(x)}{(x+1)^2}dx=\sum_{k\in N}(-1)^k k\int_0^1 x^{k-1}ln^3(x)dx=6\sum_{k\in N}\frac{(-1)^{k-1}}{k^3}=\frac{9}{2}\zeta(3)$$

$$J=\int_1^{\infty}\frac{ln^3(x)}{(x+1)^2}dx\overset{I.B.P}{\cong}-\frac{Li_3(-y)}{2(y+1)^2}\Big|_0^1+\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y(y+1)^2}dy=-\frac{Li_3(-1)}{8}-\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y+1}dy+$$
$$\int_0^1\frac{Li_2(-y)}{y}dy-\frac{1}{2}\int_0^1\frac{Li_2(-y)}{(y+1)^2}dy=\frac{3\zeta(3)}{32}-\frac{1}{2}(J_1-J_2+J_3)$$

$$\{\therefore Li_s(-1)=-\eta(s);\eta(s)=\left(1-2^{1-s}\right)\zeta s),R[s]\geq 1;Li_s(1)=\zeta s),R[s]>1\}$$

$$J_1=\int_0^1\frac{Li_2(-y)}{y+1}dy\overset{I.B.P}{=}Li_2(-1)\ln(2)+\int_0^1\frac{ln^2(y+1)}{y}dy=\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}+\int_{\frac{1}{2}}^1\frac{ln^2(y)}{y}dy+\int_{\frac{1}{2}}^1\frac{ln^2(y)}{1-y}dy=$$

$$\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}-\frac{2ln^3(2)}{3}-2\int_{\frac{1}{2}}^1 Li_2(x)\ln(x)\,dx=\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}-\frac{2ln^3(2)}{3}+2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right)+2\int_0^1 dLi_3(y)=$$

$$\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}-\frac{2ln^3(2)}{3}+2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right)+2\left(Li_3(1)-Li_3\left(\frac{1}{2}\right)\right)=\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}+\frac{\zeta(3)}{4}$$

$$J_2=\int_0^1\frac{Li_2(-y)}{y}dy=\int_0^1 dLi_3(-x)=-\frac{3\zeta(3)}{4}$$

$$J_3=\int_0^1\frac{Li_2(-y)}{(y+1)^2}dy\overset{I.B.P}{=}-\frac{Li_2(-y)}{y+1}\Bigg|_0^1$$
$$-\int_0^1\frac{\ln(y+1)}{y(y+1)}dy=\frac{Li_2(-1)}{2}+\int_0^1\frac{\ln(y+1)}{y+1}dy-\int_0^1\frac{\ln(y+1)}{y}dy=$$

$$\frac{1}{2}\cdot\frac{\pi^2}{12}+\int_0^1 d\left(\frac{ln^2(x+1)}{2}\right)-\int_0^1 d\left(-Li_2(-1)\right)=\frac{\pi^2}{24}+\frac{ln^2(2)}{2}+Li_2(-1)=\frac{ln^2(2)}{2}-\frac{\pi^2}{24}$$

$$H=\left[\frac{3\zeta(3)}{32}-\frac{1}{2}\left(\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}+\frac{\zeta(3)}{4}\right)-\frac{3\zeta(3)}{8}-\frac{1}{2}\left(\frac{ln^2(2)}{2}-\frac{\pi^2}{24}\right)\right]\cdot\frac{9}{2}\zeta(3)$$

$$\int_0^1\int_1^\infty\frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy=\frac{9}{2}\zeta(3)\left(\frac{\pi^2\ln(2)}{24}-\frac{13}{32}\zeta(3)+\frac{\pi^2}{48}-\frac{ln^2(2)}{2}\right)$$