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Prove that:

$$\int_0^1 \frac{x.\arccos(x)}{\sqrt{2-2x^4}}dx = \frac{\pi}{8\sqrt{2}}\,4F_3\left(\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},\frac{3}{2};1\right)$$

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Solution by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1 \frac{x.\arccos(x)}{\sqrt{2-2x^4}}dx=\int_0^1 \frac{x.\arccos(x)}{\sqrt{2(1-x^4)}}dx=\frac{1}{\sqrt{2}}\int_0^1 \frac{x.\arccos(x)}{\sqrt{(1-x^4)}}dx$$

$$u=\arccos(x)\quad;\quad \frac{du}{dx}=-\frac{1}{\sqrt{(1-x^2)}}$$

$$dv=\frac{x}{\sqrt{(1-x^4)}}\quad;\quad v=\frac{1}{2}\arcsin(x^2)$$

$$I=\frac{1}{\sqrt{2}}\left[\frac{1}{2}\arccos(x).\arcsin(x^2)\Big|_0^1+\frac{1}{2}\int_0^1 \frac{\arcsin(x^2)}{\sqrt{(1-x^2)}}dx\right]=\frac{1}{2\sqrt{2}}\int_0^1 \frac{\arcsin(x^2)}{\sqrt{(1-x^2)}}dx=$$

$$\underbrace{\overbrace{=}^{x^2=y,\ x=\sqrt{y}}}_{dx=\frac{dy}{2\sqrt{y}}}\frac{1}{2\sqrt{2}}\int_0^1 \frac{\arcsin(y)}{\sqrt{(1-y)}}\frac{dy}{2\sqrt{y}}=\frac{1}{4\sqrt{2}}\int_0^1 \frac{\arcsin(y)}{\sqrt{y}\sqrt{(1-y)}}dy=$$

$$\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\int_0^1 \frac{y^{2n+1}}{\sqrt{y}\sqrt{(1-y)}}dy=\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\int_0^1 y^{2n+1}.(1-y)^{-\frac{1}{2}}dy=$$

$$\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\int_0^1 y^{\frac{1}{2n+\frac{3}{2}}}.(1-y)^{-\frac{1}{2}-1}dy=\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\beta\left(2n+\frac{3}{2},\frac{1}{2}\right)=$$

$$\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2n+\frac{3}{2}+\frac{1}{2}\right)}=\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}=$$

$$\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}=\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{(n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}=$$

$$=\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{2\left(n+\frac{1}{2}\right)n!}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}$$

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This :

$$\frac{\sqrt{\pi}}{8\sqrt{2}}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{2\left(n+\frac{1}{2}\right)n!}\cdot\frac{4(n+1)\left(2n+\frac{3}{2}\right)n!}{4\left(n+\frac{3}{4}\right)(2(n+1))!}$$

$$=\frac{\sqrt{\pi}}{8\sqrt{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{4}\right)(n+1)\Gamma\left(n+\frac{5}{4}\right)}{n!\,\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)(n+1)}=$$

$$=\frac{\sqrt{\pi}}{8\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n\left(\frac{5}{4}\right)_n(1)_n}{n!\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n(1)_n}=\frac{\sqrt{\pi}}{8\sqrt{2}}4F_3\left[\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},\frac{3}{2};1\right]$$