ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \int_0^1 (x\log(\arccos(1-y)) + \arctan^2(1-y))\,dxdy$$

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$$\Omega = \int_0^1 \int_0^1 x\log(\cos^{-1}(1-y))\,dxdy + \int_0^1 \int_0^1 arctan^2(1-y)\,dxdy = \frac{1}{2}\int_0^1 \log(\cos^{-1}(y))\,dy$$

$$\text{Sub...}\left\{\arccos(y) = t\,;\ \frac{dt}{dy} = -\frac{1}{\sin(t)}\ ;\ t\left[0;\frac{\pi}{2}\right]\right\}$$

$$\Omega_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(t)\log(t) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos(t)}{t}dt + \frac{1}{2}\log(0) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos(t)-1}{t}dt + \frac{1}{2}\ln\left(\frac{\pi}{2}\right)$$

$$\left\{We\ know\ cosine\ integral\ Ci(z) = \gamma + \ln(z) + \int_0^z \frac{\cos(t)-1}{t}dz = -\int_0^\infty \frac{\cos(t)}{t}dt\right\}$$

$$\Omega_1 = \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\cos(t)-1}{t}dt + \ln\left(\frac{\pi}{2}\right)\right) = \frac{1}{2}Ci\left(\frac{\pi}{2}\right) - \frac{\gamma}{2}$$

$$\Omega_2 = \int_0^1 \int_0^1 arctan^2(1-y)\,dxdy = \int_0^1 arctan^2(y)\,dy = \Big|_0^1 yarctan^2(y) -$$

$$-2\int_0^1 \frac{yarctn(y)}{1+y^2}dy = |\arctan(y) = t| = \frac{\pi^2}{16} - \text{-}2\int_0^{\frac{\pi}{4}} t.tan(t)dt =$$

$$= \frac{\pi^2}{16} + 2\Big|_0^{\frac{\pi}{4}} t\log(\cos(t)) - 2\int_0^{\frac{\pi}{4}} \log(\cos(t))dt = C$$

$$\left\{We\ know\ Fourier\ series\ of.\ \ln(\cos(z)) = -\ln(2) - \sum_{n=1}^{\infty} \frac{(-1)^n\cos(2nz)}{n}\ \ function\right\}$$

$$\Omega_2 = \frac{\pi^2}{16} - \frac{\pi}{4}\ln(2) + \ln(2)\int_0^{\frac{\pi}{4}} dt + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} \cos(2nt)\,dt$$

$$= \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) + + \sum_{n=1}^{\infty} \frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{e^2} = \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2)$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} = \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) - G$$