ROMANIAN MATHEMATICAL MAGAZINE

**Find:**

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\log(\sin(y)+\cos(y))dy$$

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$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\log(\sin(y)+\cos(y))dy \quad \leftrightarrow \sin(y)+\cos(y)=\sqrt{2}\sin(y+\frac{\pi}{4})$$

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\sqrt{2}\sin(y+\tfrac{\pi}{4})\right)dy=\frac{1}{2}\ln(2)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} ydy+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\sin\left(x+\tfrac{\pi}{4}\right)\right)dy$$

$$\Omega_1=\frac{\ln(2)}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} ydy=\frac{\ln(2)}{2}\frac{y^2}{2}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{3\pi^2}{64}\ln(2)$$

$$\Omega_2=\frac{\ln(2)}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\sin\left(x+\frac{\pi}{4}\right)\right)dy=\left|y+\frac{\pi}{4}=x\right|=\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\left(x-\frac{\pi}{4}\right)\ln(x)dx=I_1-I_2$$

$$I_1=\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} x\ln(\sin x)dx=-\ln(2)\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} dx-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} x\cos(2nx)dx=$$

$$=-\frac{5\pi^2\ln(2)}{32}-\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\cos(2nx)}{4n^2}-\frac{x\sin(2nx)}{2n}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}=-\frac{5\pi^2\ln(2)}{32}-\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{3\pi\sin\left(\frac{3\pi n}{2}\right)}{8n^2}-\frac{x\sin(\pi n)}{4n^2}-\right.$$

$$\left.-\frac{\cos(\pi n)}{4n^3}+\frac{\cos\left(\frac{3\pi n}{2}\right)}{4n^3}\right]=-\frac{5\pi^2\ln(2)}{32}+\frac{3\pi}{8}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right)+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-$$

$$\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{3\pi n}{2}\right)}{n^3}=$$

$$=-\frac{5\pi^2\ln(2)}{32}+\frac{3\pi}{8}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\frac{1}{4}(1-2^{1-3})\zeta(3)$$

$$+\frac{1}{4}\left(-\frac{1}{2^3}+\frac{1}{4^3}-\frac{1}{6^3}+\frac{1}{8^3}-\cdots\right)=$$

$$= -\frac{5\pi^2\ln(2)}{32} + \frac{3\pi}{8}G - \frac{3}{4}\zeta(3) - \frac{1}{4}\times\frac{1}{2^3}\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots\right) =$$

$$= -\frac{5\pi^2\ln(2)}{32} + \frac{3\pi}{8}G - \frac{3}{16}\zeta(3) + \frac{1}{32}(1 - 2^{1-3})\zeta(3)$$

$$= -\frac{5\pi^2\ln(2)}{32} + \frac{3\pi}{8}G - \frac{21}{128}\zeta(3)$$

$$I_2 = \frac{\pi}{4}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \ln(\sin x)dx = -\frac{\pi}{4}\ln(2)\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} dx - \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos(2nx)dx = -\frac{\pi^2\ln(2)}{16}$$

$$- -\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\sin\left(\frac{3\pi n}{2}\right) - \sin(\pi n)}{2n}\right]$$

$$= -\frac{\pi^2\ln(2)}{16} - \frac{\pi}{8}\sum_{n=}^{\infty}\frac{\sin\left(\frac{3\pi n}{2}\right)}{n^2} + \frac{\pi}{8}\sum_{n=}^{\infty}\frac{\sin(\pi n)}{n^2} = \frac{\pi}{8}G - \frac{\pi^2\ln(2)}{16}$$

$$\Omega_2 = I_1 - I_2 = \frac{\pi}{4}G - \frac{3\pi^2\ln(2)}{32} - \frac{21}{128}\zeta(3) \qquad \Omega = \Omega_1 + \Omega_2 = \frac{\pi}{4}G - \frac{3\pi^2\ln(2)}{64} - \frac{21}{128}\zeta(3)$$