ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\int_0^1 \frac{x^2 ln^2(1+x^2)}{(1+x)(1+x^2)}dx$$

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$$I=\int_0^1 \frac{x^2 ln^2(1+x^2)}{(1+x)(1+x^2)}dx=$$

$$=\underbrace{\int_0^1 \frac{ln^2(1+x^2)}{1+x}dx}_{A}-\frac{1}{2}\underbrace{\int_0^1 \frac{ln^2(1+x^2)}{1+x^2}dx}_{B}+\frac{1}{2}\underbrace{\int_0^1 \frac{xln^2(1+x^2)}{1+x^2}dx}_{C}$$

$$\textit{Calculate the given integral's}:$$

$$A=\int_0^1 \frac{ln^2(1+x^2)}{1+x}dx\ \left\{u=ln^2(1+x^2),\frac{du}{dx}=\frac{4xln(1+x^2)}{1+x^2},v=\ln(1+x)\ dv=\frac{1}{1+x}\right\}$$

$$A=ln^2(1+x^2)\ln(1+x)|_0^1-4\underbrace{\int_0^1 \frac{xln(1+x^2)\ln(1+x)}{1+x^2}dx}_{M}=$$

$$A=ln^3(2)-4\underbrace{\int_0^1 \frac{xln(1+x^2)\ln(1+x)}{1+x^2}dx}_{M}$$

$$\underbrace{\int_0^1 \frac{xln(1+x^2)\ln(1+x)}{1+x^2}dx}_{M}=\frac{1}{2}\int_0^1 \frac{xln(1-x^2)\ln(1+x^2)}{1+x^2}dx+\frac{1}{2}\int_0^1 \frac{xln\left(\frac{1+x}{1-x}\right)\ln(1+x^2)}{1+x^2}dx$$

$$=\frac{1}{2}(A_1+A_2)$$

$$A_1=\int_0^1 \frac{xln(1-x^2)\ln(1+x^2)}{1+x^2}dx\ \left[x^2=y\,,dy=2xdx,dx=\frac{dy}{2x}\right]$$

$$\int_0^1 \frac{xln(1-x^2)\ln(1+x^2)}{1+x^2}dx=\int_0^1 \frac{xln(1-y)\ln(1+y)}{1+y}\frac{dy}{2x}=\frac{1}{2}\int_0^1 \frac{\ln(1+y)\ln(1-y)}{1+y}dy$$

$$A_1=\frac{1}{16}\zeta(3)+\frac{1}{6}ln^3(2)-\frac{\ln(2)}{4}\zeta(2),\qquad A_2=\frac{3}{8}\zeta(2)\ln(2)-\frac{21}{16}\zeta(3)+\frac{\pi}{2}G$$

$$M=\frac{1}{2}(A_1+A_2)=\frac{1}{2}\left(\frac{1}{16}\zeta(3)+\frac{1}{6}ln^3(2)-\frac{\ln(2)}{4}\zeta(2)\right)+$$

$$\frac{1}{2}\left(\frac{3}{8}\zeta(2)\ln(2)-\frac{21}{16}\zeta(3)+\frac{\pi}{2}G\right)=-\frac{5}{8}\zeta(3)+\frac{1}{12}ln^3(2)+\frac{1}{16}\zeta(2)\ln(2)+\frac{\pi}{4}G$$

$$A=ln^3(2)-4M=ln^3(2)-4\left(-\frac{5}{8}\zeta(3)+\frac{1}{12}ln^3(2)+\frac{1}{16}\zeta(2)\ln(2)+\frac{\pi}{4}G\right)$$

$$A=\frac{2}{3}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G$$

$$\int_0^1\frac{ln^2(1+x^2)}{1+x^2}dx=\frac{2}{3}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G$$

$$C=\int_0^1\frac{xln^2(1+x^2)}{1+x^2}dx$$

$$C=\frac{1}{2}ln^2(1+x^2)ln(1+x^2)\Big|_0^1-2\underbrace{\int_0^1\frac{xln^2(1+x^2)}{1+x^2}dx}_{C}=\frac{1}{2}ln^3(2)-2C$$

$$3C=\frac{1}{2}ln^3(2),\quad C=\frac{1}{6}ln^3(2),\quad C=\int_0^1\frac{xln^2(1+x^2)}{1+x^2}dx=\frac{1}{6}ln^3(2)$$

$$\underbrace{\int_0^1\frac{ln^2(1+x^2)}{1+x^2}dx}_{B}=\int_0^{\frac{\pi}{4}}\frac{ln^2\left(1+tan^2(y)\right)}{1+tan^2(y)}.sec^2(y)dy=\int_0^{\frac{\pi}{4}}\frac{ln^2\left(sec^2(y)\right)}{sec^2(y)}.sec^2(y)dy=$$

$$\int_0^{\frac{\pi}{4}}ln^2\left(sec^2(y)\right)dy=4\int_0^{\frac{\pi}{4}}ln^2(\cos(y))dy=4\int_0^{\frac{\pi}{4}}ln^2(\cos(x))dx=4\int_0^{\frac{\pi}{4}}ln^2[2\cos(x)]dx-$$

$$4ln^2(2)\int_0^{\frac{\pi}{4}}dx-2\ln(2)\int_0^{\frac{\pi}{4}}ln[\cos(x)]dx$$

Recall that :

$$ln^2[2\cos(x)]=x^2+2\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}\cos(2nx);\ ln[\cos(x)]=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}$$

$$B=4\int_0^{\frac{\pi}{4}}x^2dx+8\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx-4ln^2(2)(\frac{\pi}{4})+2ln^2(2)\int_0^{\frac{\pi}{4}}dx+$$

$$2\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx=\frac{4}{3}\left(\frac{\pi}{4}\right)^3+8\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}\left[\frac{\sin\left(\frac{\pi n}{2}\right)}{2n}\right]-$$

$$\pi ln^2(2)+2ln^2(2)\left(\frac{\pi}{4}\right)+2\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\sin\left(\frac{\pi n}{2}\right)}{2n}\right]=\frac{\pi^3}{48}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)\left[H_n-\frac{1}{n}\right]-$$

$$\frac{1}{2}\pi ln^2(2)+\ln(2)\,(-G)$$

$$B=\frac{\pi^3}{48}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)H_n-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{\pi n}{2}\right)-\frac{1}{2}\pi ln^2(2)-G\ln(2)$$

$$B=\frac{\pi^3}{48}+4S(n)-4\left[-\frac{\pi^3}{32}\right]-\frac{1}{2}\pi ln^2(2)-G\ln(2)$$

$$S(n)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)H_n, \qquad S(n)=-\int_0^1 x^{n-1}\ln(1-x)\,dx$$

$$S(n)=-\int_0^1\frac{\ln(1-x)}{x}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{\pi n}{2}\right)dx, \qquad S(n)=-\int_0^1\frac{\ln(1-x)}{x}(-\arctan(x))dx$$

$$S(n)=\int_0^1\frac{\ln(1-x)\arctan(x)}{x}dx\ \ (I.B.P), \qquad S(n)=-arctan(x)Li_2(x)\Big|_0^1+\int_0^1\frac{Li_2(x)}{1+x^2}dx$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}Li_2(x)dx, S(n)=-\frac{\pi}{4}\zeta(2)+\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2n+1}-\frac{H_{2n+1}}{(2n+1)^2}\right]$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}-\sum_{n=0}^{\infty}\frac{H_{2n+1}}{(2n+1)^2}$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\frac{\pi}{4}-\left[-\mathcal{I}Li_3(1-i)-\frac{\pi}{16}ln^2(2)-\frac{1}{2}\ln(2)\,G\right]$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\frac{\pi}{4}+\mathcal{I}Li_3(1-i)+\frac{\pi}{16}ln^2(2)+\frac{1}{2}\ln(2)\,G$$

$$B=\frac{\pi^3}{48}+4(-\frac{\pi}{4}\zeta(2)+\frac{\pi}{4}+\mathcal{I}Li_3(1-i)+\frac{\pi}{16}ln^2(2)+\frac{1}{2}\ln(2)\,G)-4\left[-\frac{\pi^3}{32}\right]-\frac{1}{2}\pi ln^2(2)$$
$$-G\ln(2)$$

$$B=-\frac{\pi^3}{48}+\pi+4\mathcal{I}Li_3(1-i)-\frac{\pi}{4}ln^2(2)+G\ln(2)$$

$$\int_0^1\frac{x^2ln^2(1+x^2)}{(1+x)(1+x^2)}dx=A-B+C=\left(\frac{2}{3}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G\right)-$$

$$-\frac{1}{2}\left(-\frac{\pi^3}{48}+\pi+4\mathcal{I}Li_3(1-i)-\frac{\pi}{4}ln^2(2)+G\ln(2)\right)+\frac{1}{2}\left(\frac{1}{6}ln^3(2)\right)$$

$$\int_0^1\frac{x^2ln^2(1+x^2)}{(1+x)(1+x^2)}dx=\frac{9}{12}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G-\frac{\pi^3}{96}-\frac{\pi}{2}-2\mathcal{I}Li_3(1-i)+$$

$$\frac{\pi}{8}ln^2(2)-\frac{G}{2}ln(2)$$