Find:

$$\Omega = \int_0^1 \frac{ln^3(x+1)}{(1+x)(3+x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\int_0^1 \frac{ln^3(x+1)}{(1+x)(3+x)}dx = \frac{1}{2}\int_0^1 \frac{ln^3(x+1)}{(1+x)}dx - \frac{1}{2}\int_0^1 \frac{ln^3(x+1)}{(3+x)}dx = \frac{1}{2}\int_0^1 d\left(\frac{1}{4}ln^4(1+x)\right) -$$

$$\frac{1}{2}\int_0^{\frac{1}{2}} \frac{ln^3(x)}{x}dx + \int_0^{\frac{1}{2}} \frac{ln^3(x)}{1+2x}dx = \frac{ln^4(2)}{8} - \frac{1}{2}\int_0^{\frac{1}{2}} d\left(\frac{1}{4}ln^4(x)\right) - \int_0^1 \frac{ln^3(x)}{1+2x}dx + \int_0^{\frac{1}{2}} \frac{ln^3(x)}{1+2x}dx =$$

$$\frac{1}{2}\int_0^1 -\frac{2ln^3(x)}{1+2x}dx + \frac{1}{2}\int_0^1 \frac{ln^3\left(\frac{x}{2}\right)}{1+x}dx = -3Li_4(-2) +$$

$$\frac{1}{2}\sum_{k=0}^{\infty}\binom{3}{k} ln^{3-k}\left(\frac{1}{2}\right)\left(\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{n-1}ln^k(x)dx\right) = -3Li_4(-2) +$$

$$\frac{1}{2}\sum_{k=0}^{\infty}\binom{3}{k} ln^{3-k}\left(\frac{1}{2}\right)\left((-1)^k k!\sum_{n\in N}\frac{(-1)^{n-1}}{n^{k+1}}\right) = -3Li_4(-2) +$$

$$\frac{1}{2}\sum_{k=0}^{\infty}\binom{3}{k} ln^{3-k}\left(\frac{1}{2}\right)\left((-1)^k k!\,\eta(k+1)\right) = -3Li_4(-2) +$$

$$\frac{1}{2}\left(ln^3\left(\frac{1}{2}\right)\ln(2) - \frac{\pi^2}{4}ln^2(2) + \frac{9}{2}\zeta(3)\ln\left(\frac{1}{2}\right) - \frac{7\pi^4}{120}\right)$$

$$\int_0^1 \frac{ln^3(x+1)}{(1+x)(3+x)}dx = -3Li_4(-2) - \frac{\pi^4}{240} - \frac{9}{4}\zeta(3)\ln(2) - \frac{\pi^2}{8}ln^2(2) - \frac{1}{2}ln^4(2)$$

Solution 2 by Pham Duc Nam-Vietnam

$$I = \int_0^1 \frac{ln^3(x+1)}{(1+x)(3+x)}dx$$

$$Let\ : x = \frac{1}{t} - 1 \to \ I = -\int_{\frac{1}{2}}^1 \frac{ln^3(t)}{2t+1}dt = -\int_{\frac{1}{2}}^1 \frac{ln^3(t)}{2t+1}d\left(\frac{1}{2}\ln(2t+1)\right) =$$

$$-\frac{1}{2}ln^4(2) + \frac{3}{2}\int_{\frac{1}{2}}^1 ln^2(t)\,d\left(-Li_2(-2t)\right) =$$

$$-\frac{1}{2}ln^4(2)+\frac{3}{2}\left(-\frac{\pi^2}{12}ln^2(2)+2\int_{\frac{1}{2}}^{1}\ln(t)\,d(Li_3(-2t)\right)=$$

$$-\frac{1}{2}ln^4(2)-\frac{\pi^2}{8}ln^2(2)+3(-\frac{3}{4}\zeta(3)\ln(2)-\int_{\frac{1}{2}}^{1}d\left(Li_4(-2t)\right)=$$

$$-\frac{1}{2}ln^4(2)-\frac{\pi^2}{8}ln^2(2)-\frac{9}{4}\zeta(3)\ln(2)-3\left(Li_4(-2)+\frac{7\pi^4}{120}\right)=$$

$$-3Li_4(-2)-\frac{7\pi^4}{240}-\frac{9}{4}\zeta(3)\ln(2)-\frac{\pi^2}{8}ln^2(2)-\frac{1}{2}ln^4(2)$$

Solution 3 by Shobhit Jain-India

$$\Omega=\int_0^1\frac{ln^3(1+x)}{(1+x)(3+x)}dx\underbrace{=}_{x\to x-1}\int_1^2\frac{ln^3x}{x(2+x)}dx=\frac{1}{2}\int_1^2\frac{ln^3x}{x}dx-\frac{1}{2}\int_1^2\frac{ln^3x}{(2+x)}dx$$

$$=\frac{ln^4 2}{8}-\frac{1}{4}\int_1^2 ln^3x\left(1+\frac{x}{2}\right)^{-1}dx=\frac{ln^4 2}{8}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\int_1^2(ln^3x)x^{n-1}dx$$

$$Now,\quad\int(ln^3x)x^{n-1}\,dx=(ln^3x)\frac{x^n}{n}-3(ln^2x)\frac{x^n}{n^2}+6(lnx)\frac{x^n}{n^3}-6\frac{x^n}{n^4}+C$$

$$\Rightarrow\int_1^2(ln^3x)x^{n-1}dx=(ln^3 2)\frac{2^n}{n}-3(ln^2 2)\frac{2^n}{n^2}+6(ln2)\frac{2^n}{n^3}-6\frac{2^n}{n^4}+\frac{6}{n^4}$$

$$\Omega=\frac{ln^4 2}{8}+\frac{1}{2}\sum_{n=1}^{\infty}\left((ln^3 2)\frac{(-1)^n}{n}-3(ln^2 2)\frac{(-1)^n}{n^2}+6(ln2)\frac{(-1)^n}{n^3}-6\frac{(-1)^n}{n^4}+\frac{6(-1)^n}{n^4 2^n}\right)$$

$$=\frac{ln^4 2}{8}-\frac{ln^4 2}{2}+\frac{3}{2}(ln^2 2)\eta(2)-3(ln2)\eta(3)+3\eta(4)+3Li_4\left(-\frac{1}{2}\right)$$

$$=\frac{ln^4 2}{8}-\frac{ln^4 2}{2}+\frac{3}{4}(ln^2 2)\zeta(2)-\frac{9}{4}(ln2)\zeta(3)+\frac{21}{8}\zeta(4)+3Li_4\left(-\frac{1}{2}\right)$$

$$=-\frac{3}{8}ln^4(2)+\frac{\pi^2}{8}\left(ln^2(2)\right)-\frac{9}{4}\ln(2)\zeta(3)+\frac{7\pi^4}{240}ln(2)+3Li_4\left(-\frac{1}{2}\right)$$

Solution 4 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^1\frac{ln^3(1+x)}{(1+x)(3+x)}dx=\frac{1}{2}\underbrace{\int_0^1\frac{ln^3(1+x)}{(1+x)}dx}_{\Omega_1}-\frac{1}{2}\underbrace{\int_0^1\frac{ln^3(1+x)}{(3+x)}dx}_{\Omega_2}$$

$$\Omega_1=\int_0^1\frac{ln^3(1+x)}{(1+x)}dx\overset{1+x\to x}{\cong}\underbrace{\int_1^2\frac{ln^3(x)}{x}dx}_{A}\overset{IBP}{\cong}ln^4(2)-3\underbrace{\int_1^2\frac{ln^3(x)}{x}dx}_{A}$$

$$A=ln^4(2)-3A\ \to\ \Omega_1=A=\frac{ln^4(2)}{4}$$

$$\Omega_2=\int_0^1\frac{ln^3(1+x)}{(3+x)}dx\overset{1+x\to x}{\cong}\int_1^2\frac{ln^3(x)}{2+x}dx=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\int_1^2 x^n\,ln^3(x)dx\overset{IBP}{\cong}$$

$$ln^3(x)\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}-3ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}+6\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}-6\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4}+$$

$$3\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^{n+1}}{(n+1)^4}=-\frac{\pi^2}{4}ln^2(2)-6Li_4\left(-\frac{1}{2}\right)+\frac{9}{2}\zeta(3)\ln(2)-\frac{7\pi^4}{240}+ln^4(2)$$

$$\Omega=\frac{1}{2}(\Omega_1-\Omega_2)=-3Li_4(-2)-\frac{7\pi^4}{240}-\frac{9}{4}\zeta(3)\ln(2)-\frac{\pi^2}{8}ln^2(2)-\frac{1}{2}ln^4(2)$$

Solution 5 by Quadri Faruk Temitope-Nigeria

$$\mathrm{I}=\int_0^1\frac{ln^3(1+x)}{(1+x)(3+x)}dx$$

$$I=\int_0^1\frac{ln^3(1+x)}{(1+x)(3+x)}dx=\frac{1}{2}\int_0^1\frac{ln^3(1+x)}{(1+x)}dx-\frac{1}{2}\int_0^1\frac{ln^3(1+x)}{(3+x)}dx$$

$$Le\ :\ 1+x=p\to\ dx=dp,\ I=\frac{1}{2}\int_1^2\frac{ln^3(p)}{(p)}dp-\frac{1}{2}\int_1^2\frac{ln^3(p)}{(p+2)}dp$$

$$I=\frac{1}{2}\cdot\frac{1}{4}ln^4(p)\Big|_1^2-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{1+n}}\int_1^2 p^n\,ln^3(p)dp,$$

$$I=\frac{1}{8}ln^4(2)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n}\int_1^2 p^{n-1}\,ln^3(p)dp$$

$$I=\frac{1}{8}ln^4(2)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\left[\frac{6}{n^4}-\frac{6.2^n}{n^4}+\frac{6.2^n\ln(2)}{n^3}-3ln^2(2)\frac{2^n}{n^2}+ln^3(2)\frac{2^n}{n}\right]$$

$$I=-3Li_4(-2)-\frac{7\pi^4}{240}-\frac{9}{4}\zeta(3)\ln(2)-\frac{\pi^2}{8}ln^2(2)-\frac{1}{2}ln^4(2)$$

Solution 6 by Obiajunwa Januarius-Nigeria

$$\int_0^1\frac{ln^3(1+x)}{(1+x)(3+x)}dx=\frac{1}{2}\int_0^1\frac{ln^3(1+x)}{(1+x)}dx-\frac{1}{2}\int_0^1\frac{ln^3(1+x)}{(3+x)}dx=$$

$$\frac{1}{2}\lim_{n\to -1}\int_0^1 (1+x)^n\, ln^3(1+x)dx - \frac{1}{4}\int_0^1 \frac{ln^3(1+x)}{1+\frac{1+x}{2}}dx =$$

$$\frac{1}{2}\lim_{n\to -1}\frac{\partial^3}{\partial n^3}\left[\int_0^1 (1+x)^n dx\right] - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\int_0^1 (1+x)^n ln^3(1+x)dx =$$

$$\frac{1}{2}\lim_{n\to -1}\frac{\partial^3}{\partial n^3}\left[\frac{(1+x)^{n+1}}{n+1}\Bigg|\begin{matrix}1\\0\end{matrix}\right] - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{\partial^3}{\partial n^3}\left[\frac{(1+x)^{n+1}}{n+1}\Bigg|\begin{matrix}1\\0\end{matrix}\right] =$$

$$\frac{1}{2}\lim_{n\to -1}\frac{\partial^3}{\partial n^3}\left[\frac{2^{n+1}}{n+1} - \frac{1}{n+1}\right] - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{\partial^3}{\partial n^3}\left[\frac{2^{n+1}}{n+1} - \frac{1}{n+1}\right] =$$

$$\frac{1}{2}\lim_{n\to 1}\left[\frac{6}{(n+1)^4} - \frac{12.2^n}{(n+1)^4} + \frac{12.\ln(2).2^n}{(n+1)^3} - \frac{6ln^2(2)2^n}{(n+1)^2} + \frac{2ln^3(2)2^n}{n+1}\right] -$$

$$\frac{6}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{1}{(n+1)^4} + \frac{12}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n}{(n+1)^4} - \frac{12}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n\ln(2)}{(n+1)^3} +$$

$$\frac{6}{4}ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n}{(n+1)^2} - \frac{2}{4}ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n}{(n+1)}$$

Extended L'Hopital's rule :

$$\lim_{n\to p}\frac{f(x)}{g(x)} = \lim_{n\to p}\frac{f^n(x)}{g^n(x)}$$

$$\Omega = \frac{1}{2}\lim_{n\to -1}\left[-\frac{12}{4!}2^n ln^4(2) + \frac{12}{3!}\ln(2).2^n ln^3(2) - \frac{6}{2!}ln^2(2)2^n ln^2(2) + 2ln^3(2)2^n\ln(2)\right] +$$

$$3\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}(n+1)^4} + 3\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4} - 3\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} +$$

$$\frac{3}{2}ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} - \frac{ln^3(2)}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \frac{1}{2}[-\frac{ln^4(2)}{4} + ln^4(2) - \frac{3}{3}ln^4(2) + ln^4(2)] +$$

$$3Li_4\left(-\frac{1}{2}\right) + 3\eta(4) - 3\ln(2)\,\eta(3) + \frac{3}{2}ln^2(2)\eta(2) - \frac{ln^4(2)}{2}$$

$$\Omega = \frac{ln^4(2)}{8} + 3Li_4\left(-\frac{1}{2}\right) + \frac{21}{8}\zeta(4) - \frac{9}{4}\ln(2)\,\zeta(3) + \frac{3}{4}ln^2(2)\zeta(2) - \frac{ln^4(2)}{2}$$

$$\Omega = -3Li_4(-2) - \frac{7\pi^4}{240} - \frac{9}{4}\zeta(3)\ln(2) - \frac{\pi^2}{8}ln^2(2) - \frac{1}{2}ln^4(2)$$