ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\Omega=\int_0^{\frac{1}{2}} x^3\psi(1+x^2)dx=\frac{1}{16}\left(2+ln\left(\frac{8\Gamma^2\left(\frac{5}{4}\right)}{\pi A^9}\right)-\frac{2G}{\pi}\right)$$

$A = Glaisher - Kinkelin's\ constant, G = Catalan's\ constant,$

$\psi(x) = digamma\ function$

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$$\Omega=\int_0^{\frac{1}{2}} x^3\psi(1+x^2)dx=\int_{x=0}^{\frac{1}{2}} x^2\psi(1+x^2)xdx \underbrace{=}_{\substack{x\to\sqrt{u}\\u=x^2}} \frac{1}{2}\int_{u=0}^{\frac{1}{4}} u\,\psi(1+u)du=\frac{1}{2}\int_{u=0}^{\frac{1}{4}} u\frac{\Gamma'(1+u)}{\Gamma(1+u)}du$$

$$=\left[\frac{u}{2}ln\Gamma(1+u)\right]_0^{\frac{1}{4}}-\frac{1}{2}\int_{u=0}^{\frac{1}{4}} ln\Gamma(1+u)du=\frac{1}{8}ln\Gamma\left(\frac{5}{4}\right)-\frac{1}{2}I$$

$Here,\ \ I=\int_0^{\frac{1}{4}} ln\Gamma(1+x)dx\,.Now\ we\ can\ use\ Kummer's\ series\ for\ the\ function\ ln\Gamma(x)$

$$\Rightarrow\frac{1}{2}ln\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)=\left(\frac{1}{2}-x\right)(\gamma+ln2\pi)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log_e n}{n}sin(2n\pi x)\quad for\ 0<x<1$$

$(\gamma = Euler - Mascheroni\ \ Constant)$

$$\frac{\Gamma(x)}{\Gamma(1-x)}=\frac{\Gamma^2(x+1)2\,sin(\pi x)}{2\pi x^2}$$

$$\Rightarrow\frac{1}{2}ln\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)=ln\Gamma(1+x)+\frac{1}{2}ln(2\,sin(\pi x))-\frac{1}{2}ln(2\pi)-lnx$$

$$\Rightarrow ln\Gamma(1+x)=\frac{1}{2}ln(2\pi)+\left(\frac{1}{2}-x\right)(\gamma+ln2\pi)-\frac{1}{2}ln(2\,sin(\pi x))+lnx+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log_e n}{n}sin(2n\pi x)$$

$$\Rightarrow I=\frac{ln(2\pi)}{8}+(\gamma+ln2\pi)\underbrace{\int_0^{\frac{1}{4}}\left(\frac{1}{2}-x\right)dx}_{P}-\frac{1}{2}\underbrace{\int_0^{\frac{1}{4}}ln(2\sin(\pi x))dx}_{Q}+\underbrace{\int_0^{\frac{1}{4}}lnxdx}_{R}$$
$$+\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{log_e n}{n^2}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$$P=\int_0^{\frac{1}{4}}\left(\frac{1}{2}-x\right)dx\underbrace{=}_{x\to\frac{1}{4}-x}\int_0^{\frac{1}{4}}\left(x+\frac{1}{4}\right)dx=\frac{1}{2}\times\left(\frac{1}{2}+\frac{1}{4}\right)\times\frac{1}{4}=\frac{3}{32}$$

$$Q=\int_0^{\frac{1}{4}}ln(2\sin(\pi x))dx\underbrace{=}_{x\to\frac{x}{2\pi}}\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}ln\left(2\sin\left(\frac{x}{2}\right)\right)dx=\frac{-1}{2\pi}\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}}\frac{\cos(nx)}{n}dx=$$
$$=\frac{-1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}=-\frac{G}{2\pi}$$

$$R=\int_0^{\frac{1}{4}}lnxdx=[xlnx-x]_0^{\frac{1}{4}}=-\frac{1}{4}-\frac{ln2}{2}$$

$$\Rightarrow I=\frac{ln(2\pi)}{8}+\frac{3}{32}(\gamma+ln2\pi)+\frac{G}{4\pi}-\frac{1}{4}-\frac{ln2}{2}+\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{log_e n}{n^2}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$$Let,\ \ N=\sum_{n=1}^{\infty}\frac{log_e n}{n^2}=O+E\ \ where,\ \ O=\sum_{n=1}^{\infty}\frac{log_e(2n-1)}{(2n-1)^2}\ and\ E=\sum_{n=1}^{\infty}\frac{log_e(2n)}{(2n)^2}$$

$$\Rightarrow\ \ E=\frac{ln2}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{log_e n}{n^2}=\frac{\pi^2 ln2}{24}+\frac{N}{4}\Rightarrow\ \ O=N-E=\frac{3N}{4}-\frac{\pi^2 ln2}{24}$$

$$\Rightarrow\ \ O-E=\left(\frac{3N}{4}-\frac{\pi^2 ln2}{24}\right)-\left(\frac{\pi^2 ln2}{24}+\frac{N}{4}\right)=\frac{N}{2}-\frac{\pi^2 ln2}{12}$$

$$Now,\ \sum_{n=1}^{\infty}\frac{log_e n}{n^2}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)=\left(\frac{ln1}{1^2}+\frac{ln3}{3^2}+\frac{ln5}{5^2}\dots\dots\right)+2\left(\frac{ln2}{2^2}+\frac{ln6}{6^2}+\frac{ln10}{10^2}\dots\dots\right)$$

$$=\underbrace{\left(\frac{ln1}{1^2}+\frac{ln3}{3^2}+\frac{ln5}{5^2}\dots\dots\right)}_{O}+\underbrace{\left(\frac{ln2}{2^2}+\frac{ln4}{4^2}+\frac{ln6}{6^2}\dots\dots\right)}_{E}+\left(\frac{ln2}{2^2}-\frac{ln4}{4^2}+\frac{ln6}{6^2}-\ \dots\dots\right)$$

$$=N+\frac{ln2}{2^2}\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ \dots\dots\right)+\frac{1}{2^2}\left(\frac{ln1}{1^2}-\frac{ln2}{2^2}+\frac{ln3}{3^2}-\ \dots\dots\right)=N+\frac{\pi^2 ln2}{48}+\frac{1}{4}(O-E)=$$

$$= N + \frac{\pi^2 ln2}{48} + \frac{1}{4}\left(\frac{N}{2} - \frac{\pi^2 ln2}{12}\right) = \frac{9}{8}N$$

$$\Rightarrow \ I = \frac{ln(2\pi)}{8} + \frac{3}{32}(\gamma + ln2\pi) + \frac{G}{4\pi} - \frac{1}{4} - \frac{ln2}{2} + \frac{9}{8}\left(\frac{N}{2\pi^2}\right)$$

$$Now, \ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ \Rightarrow \zeta'(s) = -\sum_{n=1}^{\infty} \frac{log_e n}{n^s} \ \Rightarrow \zeta'(2) = -\sum_{n=1}^{\infty} \frac{log_e n}{n^2} = -N$$

$$Now, \ \ -\frac{\zeta'(2)}{\zeta(2)} = 12lnA - \gamma - ln(2\pi)$$

$$\Rightarrow N = \ -\zeta'(2) = \frac{\pi^2}{6}(12lnA - \gamma - ln(2\pi)) \Rightarrow \frac{N}{2\pi^2} = lnA - \frac{\gamma}{12} - \frac{ln(2\pi)}{12}$$

$$\Rightarrow \ I = \frac{ln(2\pi)}{8} + \frac{3}{32}(\gamma + ln2\pi) + \frac{G}{4\pi} - \frac{1}{4} - \frac{ln2}{2} + \frac{9}{8}\left(lnA - \frac{\gamma}{12} - \frac{ln(2\pi)}{12}\right)$$

$$= \frac{ln(2\pi)}{8} + \frac{G}{4\pi} - \frac{1}{4} - \frac{ln2}{2} + \frac{9}{8}lnA \Rightarrow \ I = \frac{ln\,\pi}{8} + \frac{G}{4\pi} - \frac{1}{4} - \frac{3ln2}{8} + \frac{9}{8}lnA$$

$$\Rightarrow \Omega = \frac{1}{8}ln\Gamma\left(\frac{5}{4}\right) - \frac{1}{2}\left(\frac{ln\,\pi}{8} + \frac{G}{4\pi} - \frac{1}{4} - \frac{3ln2}{8} + \frac{9}{8}lnA\right) =$$

$$= \frac{1}{16}\left(2 - \frac{2G}{\pi} + 2ln\Gamma\left(\frac{5}{4}\right) + 3ln2 - ln\pi - 9lnA\right) \Rightarrow \Omega = \frac{1}{16}\left(2 + ln\left(\frac{8\Gamma^2\left(\frac{5}{4}\right)}{\pi A^9}\right) - \frac{2G}{\pi}\right)$$