ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\Omega=\int_0^\infty\frac{x\ln^2(1+x)}{(1+x)(2+x)(3+x)}dx=|1+x=t|=\int_1^\infty\frac{(t-1)\ln^2(t)}{t(t+1)(t+2)}dt$$

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$$\text{Replace }\left\{\frac{1}{t}=u\quad \frac{du}{dt}=-u^2\quad u[0;1]\right\}$$

$$\Omega=\int_0^1\frac{\left(\frac{1}{u}-1\right)\ln^2(u)}{\frac{1}{u}\left(\frac{1}{u}+1\right)\left(\frac{1}{u}+2\right)}\frac{du}{u^2}=\int_0^1\frac{(1-u)\ln^2(u)}{(1+u)(1+2u)}du=2\int_0^1\frac{(1-u)\ln^2(u)}{(1+2u)}du-\int_0^1\frac{(1-u)\ln^2(u)}{(1+u)}du=$$

$$=2\Omega_1-\Omega_2$$

$$\Omega_1=\int_0^1\frac{\ln^2(u)}{(1+2u)}du-\int_0^1\frac{u\ln^2(u)}{(1+2u)}du=\frac{1}{2}\left(\int_0^1\frac{\ln^2(u)}{\left(u+\frac{1}{2}\right)}du-\int_0^1\frac{u\ln^2(u)}{\left(u+\frac{1}{2}\right)}du\right)$$

$$\text{General solution: } I(a)=\int\frac{\ln^2(x)}{x+a}dx\qquad \text{Replace: }\left\{x=at\quad \frac{dx}{dt}=a\right\}$$

$$⍟I(a)=\int\frac{\ln^2(at)}{1+t}dt=\ln^2(at)\ln(1+t)-2\int\frac{\ln(at)\ln(1+t)}{t}dt=\ln^2(at)\ln(1+t)+$$

$$+2\sum_{n=1}^\infty\frac{(-1)^n}{n}\int t^{n-1}(\ln(a)+\ln(t))\,dt=\ln^2(at)\ln(1+t)+2Li_2(-t)\ln(at)-$$

$$-2Li_3(-t)\quad+c\ =\ln^2(x)\ln\left(1+\frac{x}{a}\right)+2Li_2\left(-\frac{x}{a}\right)\ln(x)-2Li_3\left(-\frac{x}{a}\right)+c$$

$$⍟J(a)=a\int\frac{\ln^2(at)}{1+t}dt=at\ln^2(at)-a\ln(1+t)\ln^2(at)-2a\int\frac{\ln(at)(t-\ln(1+t))}{t}dt=$$

$$=at\ln^2(at)-a\ln(1+t)\ln^2(at)-2a(\ln(at)-1)$$

$$-2a\sum_{n=1}^\infty\frac{(-1)^n}{n}\int t^{n-1}(\ln(a)+\ln(t))\,dt$$

$$=at\ln^2(at)-a\ln(1+t)\ln^2(at)+2at-2at\ln(at)+2Li_3(-t)-2aLi_2(-t)\ln^2(at)+c=$$

$$=2aLi_3\left(-\frac{x}{a}\right)-2aLi_2\left(-\frac{x}{a}\right)\ln(x)-a\ln^2(x)\ln\left(1+\frac{x}{a}\right)+2x+x\ln^2(x)$$

$$-2x\ln(x)+c$$

$$\blacklozenge\Omega_1=\frac{1}{2}\left(\Big|_0^1 I\left(\frac{1}{2}\right)-J\left(\frac{1}{2}\right)\right)=\frac{1}{2}(-2Li_3(-2)-Li_3(-2)-2)=-\frac{3}{2}Li_3(-2)-1$$

$$\blacklozenge\Omega_2=\int_0^1\frac{(1-u)\ln^2(u)}{(1+u)}du=\int_0^1\frac{\ln^2(u)}{1+u}du-\int_0^1\frac{u\ln^2(u)}{1+u}du=$$

$$= \sum_{n=1}^{\infty}(-1)^n \int_0^1 u^2 \ln^2(u)du$$

$$- \int_0^1 u^{n+1}\ln^2(u)du = \sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{(n+1)^3} - \frac{2}{(n+2)^3}\right) =$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^3} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = \frac{3}{2}\zeta(3) - 2 + \frac{3}{2}\zeta(3) = 3\zeta(3) - 2$$

$$\Omega = \int_0^{\infty}\frac{x\ln^2(1+x)}{(1+x)(2+x)(3+x)}dx = 2\Omega_1 - \Omega_2 = -3Li_3(-2) - 3\zeta(3)$$

Note; $\zeta(3)$ − Apery's constant