ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$\int_0^1\int_0^\infty \frac{\log^2(x)\log^2(1+y^2)}{y(1+x^2)}dx\,dy = \frac{\pi^3}{64}\zeta(3)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I = \int_0^1\int_0^\infty \frac{\log^2(x)\log^2(1+y^2)}{y(1+x^2)}dx\,dy = \int_0^\infty \frac{\log^2(x)}{1+x^2}dx\int_0^1 \frac{\log^2(1+y^2)}{y}dy$$

$$I_1 = \int_0^\infty \frac{\log^2(x)}{1+x^2}dx = \int_0^1 \frac{\log^2(x)}{1+x^2}dx + \int_1^\infty \frac{\log^2(x)}{1+x^2}dx = 2\int_0^1 \frac{\log^2(x)}{1+x^2}dx$$

$$= 2\sum_{k=0}^\infty (-1)^k \int_0^1 x^{2k}\log^2(x)\,dx = 4\sum_{k=0}^\infty \frac{(-1)^k}{(1+2k)^3} = \frac{\pi^3}{8}$$

$$I_2 = \int_0^1 \frac{\log^2(1+y^2)}{y}dy\Bigg]_{y^2=z} = \frac{1}{2}\int_0^1 \frac{\log^2(1+z)}{z}dz \overset{IBP}{=} -\int_0^1 \frac{\log(x)\log(1+x)}{1+x}dx$$

$$= \frac{1}{8}\zeta(3)$$

$$I = I_1 I_2 = \frac{\pi^3}{64}\zeta(3)$$

NOTE:

$$\int_0^1 \frac{\log(x)\log(1+x)}{1+x}dx = -\frac{1}{8}\zeta(3) \text{ and } \sum_{k=0}^\infty \frac{(-1)^k}{(1+2k)^3} = \frac{\pi^3}{32}$$