ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\mathbf{I}=\int_0^\infty \frac{\log(x)}{(1+x)(1+\mathrm{x}^2)}\mathrm{dx}$$

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$$\mathbf{I}=\int_0^\infty \frac{\log(x)}{(1+x)(1+\mathrm{x}^2)}\mathrm{dx}=\int_0^1 \frac{\log(x)}{(1+x)(1+\mathrm{x}^2)}\mathrm{dx}+\left.\int_1^\infty \frac{\log(x)}{(1+x)(1+\mathrm{x}^2)}\mathrm{dx}\right]_{x\to\frac{1}{x}}$$

$$=\int_0^1 \frac{\log(x)}{(1+x)(1+\mathrm{x}^2)}\mathrm{dx}-\int_0^1 \frac{x\log(x)}{(1+x)(1+\mathrm{x}^2)}\mathrm{dx}$$

$$=2\int_0^1 \frac{\log(x)}{(1+x)(1+\mathrm{x}^2)}\mathrm{dx}-\int_0^1 \frac{\log(x)}{1+x^2}\mathrm{dx}$$

$$=\int_0^1 \frac{\log(x)}{1+x}\mathrm{dx}-\left.\int_0^1 \frac{x\log(x)}{1+x^2}\mathrm{dx}\right]_{x\to\frac{1}{x}}=\int_0^1 \frac{\log(x)}{1+x}\mathrm{dx}-\frac{1}{4}\int_0^1 \frac{\log(x)}{1+x}\mathrm{dx}$$

$$=\frac{3}{4}\int_0^1 \frac{\log(x)}{1+x}\mathrm{dx}=\frac{3}{4}\sum_{k=0}^\infty(-1)^k\int_0^1 x^k\log(x)\,dx=-\frac{3}{4}\sum_{k=0}^\infty\frac{(-1)^k}{(k+1)^2}$$

$$=-\frac{3}{4}\eta(2)=-\frac{3}{8}\varsigma(2)=-\frac{\pi^2}{16}$$