ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\int_0^\infty \frac{\tan^{-1}(x)}{x(1+x)^3}dx$$

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$$I=\int_0^\infty \frac{\tan^{-1}(x)}{x(1+x)^3}dx=\int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^3}dx+\int_1^\infty \frac{\tan^{-1}(x)}{x(1+x)^3}dx\Bigg]_{x\to 1/x}$$

$$=\int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^3}dx+\int_0^1 \frac{x^2\tan^{-1}\left(\frac{1}{x}\right)}{(1+x)^3}dx$$

$$=\int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^3}dx-\int_0^1 \frac{x^2\tan^{-1}(x)}{(1+x)^3}dx+\frac{\pi}{2}\int_0^1 \frac{x^2}{(1+x)^3}dx$$

$$=\int_0^1 \tan^{-1}(x)\left\{\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}\right\}dx$$

$$-\int_0^1 \tan^{-1}(x)\left\{\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}\right\}dx+\frac{\pi}{2}\int_0^1 \frac{x^2}{(1+x)^3}dx$$

$$=\int_0^1 \frac{\tan^{-1}(x)}{x}dx-2\int_0^1 \frac{\tan^{-1}(x)}{x+1}dx+\int_0^1 \frac{\tan^{-1}(x)}{(x+1)^2}dx-2\int_0^1 \frac{\tan^{-1}(x)}{(x+1)^3}dx$$

$$+\frac{\pi}{2}\int_0^1 \frac{x^2}{(1+x)^3}dx$$

$$=G-2\left\{\frac{\pi}{8}\log(2)\right\}+\left\{\frac{\log(2)}{4}\right\}-2\left\{\frac{\log(2)}{8}-\frac{\pi}{32}+\frac{1}{8}\right\}+\frac{\pi}{2}\left\{\log(2)-\frac{5}{8}\right\}$$

$$=G+\frac{\pi}{4}\log(2)-\frac{\pi}{4}-\frac{1}{4}$$