ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\int_1^{\infty}\frac{x\ln(x)}{x^4+x^2+1}dx$$

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$$I=\int_1^{\infty}\frac{x\ln(x)}{x^4+x^2+1}dx\Bigg]_{x^2\to 1/x}=-\frac{1}{4}\int_0^1\frac{\ln(x)}{x^2+x+1}dx=-\frac{1}{4}I_1$$

$$I_1=\int_0^1\frac{\ln(x)}{x^2+x+1}dx=\int_0^1\frac{(1-x)\ln(x)}{1-x^3}dx=-\int_0^{\infty}\frac{y(1-e^{-y})e^{-y}}{1-e^{-3y}}dy$$

$$=-\sum_{k=0}^{\infty}\int_0^{\infty}y(1-e^{-y})e^{-y}e^{-3ky}\,dy$$

$$=-\sum_{k=0}^{\infty}\int_0^{\infty}ye^{-(3k+1)y}\,dy+\sum_{k=0}^{\infty}\int_0^{\infty}ye^{-(3k+2)y}\,dy$$

$$=-\sum_{k=0}^{\infty}\frac{\Gamma(2)}{(3k+1)^2}+\sum_{k=0}^{\infty}\frac{\Gamma(2)}{(3k+2)^2}$$

$$=-\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2}+\sum_{k=0}^{\infty}\frac{1}{(3k+2)^2}=-\frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{3}\right)^2}+\frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{2}{3}\right)^2}$$

$$=-\frac{1}{9}\varphi'\left(\frac{1}{3}\right)+\frac{1}{9}\varphi'\left(\frac{2}{3}\right)=-\frac{1}{9}\varphi'\left(\frac{1}{3}\right)+\frac{1}{9}\left(\frac{4\pi^2}{3}-\varphi'\left(\frac{1}{3}\right)\right)=\frac{4\pi^2}{27}-\frac{2}{9}\varphi'\left(\frac{1}{3}\right)$$

$$=\frac{2}{9}\left(\frac{2\pi^2}{3}-\varphi'\left(\frac{1}{3}\right)\right)$$

$$I=-\frac{1}{4}I_1=-\frac{1}{4}\left\{\frac{2}{9}\left(\frac{2\pi^2}{3}-\varphi'\left(\frac{1}{3}\right)\right)\right\}=\frac{1}{18}\varphi'\left(\frac{1}{3}\right)-\frac{\pi^2}{27}$$