ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^\infty \frac{\log(1+x)}{x(1+x)^4}dx=\varsigma(2)-\frac{49}{36}$$

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Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^\infty \frac{\log(1+x)}{x(1+x)^4}dx\Bigg]_{1+x=y}=\int_1^\infty \frac{\log(y)}{(y-1)y^4}dy\Bigg]_{\frac{1}{y}=z}=-\int_0^1 \frac{z^3\log(z)}{1-z}dz$$

$$=-\sum_{k=0}^{\infty}\int_0^1 z^{k+3}\log(z)\,dz=$$

$$=-\sum_{k=0}^{\infty}\left\{-\frac{1}{(k+4)^2}\right\}=\sum_{k=0}^{\infty}\frac{1}{(k+4)^2}=\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}-\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right)=\varsigma(2)-\frac{49}{36}$$

$$\text{Note: }\int_0^1 x^m\log^n(x)\,dx=(-1)^n\frac{n!}{(m+1)^{n+1}}$$