ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_1^{\infty}\frac{x\sqrt{x}\log(x)}{x^3+x\sqrt{x}+1}dx=\frac{4}{81}\left(\varphi_1\left(\frac{1}{9}\right)-\varphi_1\left(\frac{4}{9}\right)\right)$$

where $\varphi_1(s)$ denotes trigamma function

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$$\text{Let } x^3=m$$

$$I=\frac{4}{9}\int_1^{\infty}\frac{m^{\frac{2}{3}}\log(m)}{m^2+m+1}dm=\frac{4}{9}\int_1^{\infty}\frac{m^{\frac{2}{3}}(1-m)\log(m)}{1-m^3}dm$$

$$\text{Let } m=1/z$$

$$I=-\frac{4}{9}\int_0^1\frac{z^{-\frac{2}{3}}(z-1)\log(z)}{z^3-1}dz=-\frac{4}{9}\int_0^1\frac{z^{\frac{1}{3}}\log(z)}{z^3-1}dz+\frac{4}{9}\int_0^1\frac{z^{-\frac{2}{3}}\log(z)}{z^3-1}dz$$

$$=\frac{4}{9}\int_0^1\frac{z^{\frac{1}{3}}\log(z)}{1-z^3}dz-\frac{4}{9}\int_0^1\frac{z^{-\frac{2}{3}}\log(z)}{1-z^3}dz$$

$$=\frac{4}{9}\sum_{k=0}^{\infty}\int_0^1 z^{3k+\frac{1}{3}}\log(z)\,dz-\frac{4}{9}\sum_{k=0}^{\infty}\int_0^1 z^{3k-\frac{2}{3}}\log(z)\,dz$$

$$=-\frac{4}{9}\sum_{k=0}^{\infty}\frac{1}{\left(3k+\frac{4}{3}\right)^2}+\frac{4}{9}\sum_{k=0}^{\infty}\frac{1}{\left(3k+\frac{1}{3}\right)^2}$$

$$=-\frac{4}{81}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{4}{9}\right)^2}+\frac{4}{81}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{9}\right)^2}=\frac{4}{81}\left(\varphi_1\left(\frac{1}{9}\right)-\varphi_1\left(\frac{4}{9}\right)\right)$$