If $x \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(\int_0^1 \left(1 - t^{\frac{1}{\sin^2 x}}\right) dt\right)\left(\int_0^1 \left(1 - t^{\frac{1}{\cos^2 x}}\right) dt\right) \le \int_0^1 (1 - t)\, dt$$

Proposed by Daniel Sitaru – Romania

Solution by Samir HajAli-Damascus-Syria

First, let's consider a function f defined on $\left(0, \frac{\pi}{2}\right)$

$$f: x \to f(x) = \frac{1}{1 + \cos^2 x} \cdot \frac{1}{1 + \sin^2 x}$$

$$f'(x) = \frac{+2\cos x \sin x}{(1 + \cos^2 x)^2} \cdot \frac{1}{1 + \sin^2 x} + \frac{-2\sin x \cdot \cos x}{(1 + \sin^2 x)^2} \cdot \frac{1}{1 + \cos^2 x}$$

$$= \frac{\sin 2x}{(1 + \cos^2 x)(1 + \sin^2 x)}\left(\frac{1}{1 + \cos^2 x} - \frac{1}{1 + \sin^2 x}\right)$$

$$= \frac{\sin 2x}{(1 + \cos^2 x)(1 + \sin^2 x)}\left(\frac{\sin^2 x - \cos^2 x}{(1 + \cos^2 x)(1 + \sin^2 x)}\right)$$

Now:

$$f'(x) = 0 \Rightarrow \cos^2 x - \sin^2 x = 0$$

$$\Rightarrow \cos(2x) = 0 \Rightarrow 2x = \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \Rightarrow x = \frac{\pi}{4} + \frac{\pi k}{2}$$

For $k = 0 \Rightarrow x = \frac{\pi}{4} \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{1}{1+\frac{1}{2}} \cdot \frac{1}{1+\frac{1}{2}} = \frac{4}{5}$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$f'(x)$	$- - - -$	0	$+ + + + + +$
$f(x)$	$\frac{1}{2}$ ↘	$\frac{4}{9}$ ↗	$\frac{1}{2}$

$$\Rightarrow \text{For } x \in \left(0, \frac{\pi}{2}\right), f(x) < \frac{1}{2}$$

$$\left(\int_0^1 \left(1 - t^{\frac{1}{\sin^2 x}}\right) dt\right)\left(\int_0^1 \left(1 - t^{\frac{1}{\cos^2 x}}\right) dt\right) \le \int_0^1 (1 - t)\, dt$$

$$\left(1-\frac{t^{\frac{1}{\sin^2 x}+1}}{\frac{t}{\sin^2 x}+1}\Bigg|_0^1\right)\left(1-\frac{t^{\frac{1}{\cos^2 x}+1}}{\frac{1}{\cos^2 x}+1}\Bigg|_0^1\right)\leq 1-\frac{t^2}{2}\Bigg|_0^1$$

$$\Leftrightarrow\left(1-\frac{1}{\frac{1}{\sin^2 x}+1}\right)\left(1-\frac{1}{\frac{1}{\cos^2 x}+1}\right)\leq 1-\frac{1}{2}$$

$$\Leftrightarrow\left(1-\frac{\sin^2 x}{1+\sin^2 x}\right)\left(1-\frac{\cos^2 x}{1+\cos^2 x}\right)\leq\frac{1}{2}\Leftrightarrow\left(\frac{1}{1+\sin^2 x}\right)\left(\frac{1}{1+\cos^2 x}\right)\leq\frac{1}{2}$$

$$\Leftrightarrow f(x)\leq\frac{1}{2}\Leftrightarrow x\in\left(0,\frac{\pi}{2}\right)$$