ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \leq b$ then:

$$\int_a^b e^{x^2}\,dx \geq (b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

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$$e^x \geq x+1, (\forall)x>0 \Rightarrow e^{x^2} \geq x^2+1$$

$$\Rightarrow \int_a^b e^{x^2}\,dx \geq \int_a^b (x^2+1)\,dx = \frac{b^3-a^3}{3}+b-a =$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3}+b-a = (b-a)\cdot\left(\frac{a^2+ab+b^2}{3}+1\right) \geq$$

$$\geq (b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

$$b-a\geq 0, \frac{a^2+ab+b^2}{3}+1 \geq \sqrt[3]{a^2+ab+b^2}, \qquad S=a^2+ab+b^2>0$$

$$\frac{S+3}{3} \geq \sqrt[3]{S} \Rightarrow (S+3)^3 \geq 27S \Rightarrow S^3+9S^2+27S-27S \geq 0$$

$\Rightarrow S^3+9S^2>0,$ true, $(\forall)S>0$. Then: $\int_a^b e^{x^2}\,dx \geq (b-a)\sqrt[3]{a^2+ab+b^2}$

Equality holds for $a=b$.