ROMANIAN MATHEMATICAL MAGAZINE

If $a,b\in\mathbb{R}, a\le b, f:[a,b]\to(0,\infty), f-$ continuous then:

$$3\int_a^b f(x)\,dx+\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge 4(b-a)^{\frac{3}{2}}$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

The function $\ln$ is concave so:

$$\ln\left(\frac{3}{4}\int_a^b f(x)\,dx+\frac{1}{4}\left(\int_a^b\frac{1}{f(x)}dx\right)^3\right)\ge\frac{3}{4}\ln\left(\int_a^b f(x)\,dx\right)+\frac{1}{4}\ln\left(\left(\int_a^b\frac{1}{f(x)}dx\right)^3\right)$$

i.e.:

$$\ln\left(\frac{3}{4}\int_a^b f(x)\,dx+\frac{1}{4}\left(\int_a^b\frac{1}{f(x)}dx\right)^3\right)\ge\frac{3}{4}\ln\left(\left(\int_a^b f(x)\,dx\right)\left(\int_a^b\frac{1}{f(x)}dx\right)\right)$$

So (since exp is growing):

$$\frac{3}{4}\int_a^b f(x)\,dx+\frac{1}{4}\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge\left(\left(\int_a^b f(x)\,dx\right)\left(\int_a^b\frac{1}{f(x)}dx\right)\right)^{\frac{3}{4}}$$

The Cauchy – Schwarz inequality gives:

$$\left(\int_a^b\left(\sqrt{f(x)}\right)^2dx\right)\left(\int_a^b\left(\frac{1}{\sqrt{f(x)}}\right)^2dx\right)\ge\left(\int_a^b\sqrt{f(x)}\frac{1}{\sqrt{f(x)}}dx\right)^2$$

i.e.:

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b\frac{1}{f(x)}dx\right)\ge(b-a)^2$$

So:

$$\frac{3}{4}\int_a^b f(x)\,dx+\frac{1}{4}\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge(b-a)^{\frac{3}{2}}$$

Finally:

$$3\int_a^b f(x)\,dx+\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge 4(b-a)^{\frac{3}{2}}$$

Equality holds for $a=b$.