ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \leq b \leq \pi$ then:

$$\int_a^b \frac{\sin x}{x}dx + \int_a^b \frac{\cos x}{x^2}dx \leq \frac{2(b-a)}{ab}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban-Romania

$$\left(\frac{\cos x}{x}\right)' = \frac{-x\sin x - \cos x}{x^2} = -\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)$$

$$\int_a^b \frac{\sin x}{x}dx + \int_a^b \frac{\cos x}{x^2}dx = -\frac{\cos x}{x}\Big|_a^b =$$

$$= -\frac{\cos b}{b} + \frac{\cos a}{a} = \frac{b\cos a - a\cos b}{ab} \leq \frac{2(b-a)}{ab}, ab > 0 \Rightarrow b\cos a - a\cos b$$

$$\leq 2b - 2a \Rightarrow 2a - a\cos b \leq 2b - b\cdot\cos a$$

$$a(2-\cos b) \leq b(2-\cos a), 2-\cos \alpha > 0$$

because $\cos\alpha \leq 1 < 2 \Rightarrow 2-\cos\alpha > 0 \Rightarrow \frac{2-\cos b}{b} \leq \frac{2-\cos a}{a}, f:(0,\pi] \to \mathbb{R}$

$$f(x) = \frac{2-\cos x}{x}, f(b) \leq f(a), a \leq b$$

$$f'(x) = \frac{x\sin x - 2 + \cos x}{x^2}; \varphi:(0,\pi] \to \mathbb{R}$$

$$\varphi(x) = x\sin x - 2 + \cos x, \varphi'(x) = \sin x + x\cos x - \sin x, \varphi'(x) = x\cos x,$$

$$\varphi'(x) = 0 \Rightarrow x = \frac{\pi}{2}$$

x	0 $\frac{\pi}{2}$ π
$f'(x)$	$+++++++0------$
$f(x)$	-1 ↗ $\frac{\pi}{2}-2<0$ ↘ -3

$$\Rightarrow \varphi(x) < 0; (\forall)x \in (0,\pi] \Rightarrow f'(x) < 0, (\forall)x \in (0,\pi] \Rightarrow f \searrow (0,\pi]$$

$a \leq b \Rightarrow f(a) \geq f(b)$, true.

Then

$$\int_a^b \frac{\sin x}{x}dx + \int_a^b \frac{\cos x}{x^2}dx \leq \frac{2(b-a)}{ab}, \qquad (\forall)0 < a \leq b \leq \pi$$

Solution 2 by Adrian Popa-Romania

$$\frac{2(b-a)}{ab} = 2\left(\frac{1}{a}-\frac{1}{b}\right) = 2\frac{1}{x}\Big|_a^b = 2\int_a^b \frac{1}{x^2}dx$$

So we must show that $x\sin x + \cos x \le 2; (\forall)x \in (0,\pi)$

$$f(x) = x\sin x + \cos x - 2$$

$$f'(x) = \sin x + x\cos x - \sin x = x\cos x = 0 \Rightarrow \begin{matrix} x = 0 \\ x = \frac{\pi}{2} \end{matrix}$$

x	0	$\frac{\pi}{2}$	π
$f'(x)$	$+ + + + + + +$	0	$- - - - - - -$
$f(x)$	-1 ↗	$\frac{\pi}{2} - 2 < 0$	↘ -3

$$\Rightarrow f(x) < 0 \ \ (\forall)x \in [0,\pi) \Rightarrow$$

$$\Rightarrow x\sin x + \cos x < 2 \Rightarrow \int_a^b \frac{\sin x}{x}dx + \int_a^b \frac{\cos x}{x^2}dx \le \frac{2(b-a)}{ab}$$

Solution 3 by Khaled Abd Imouti-Syria

$$\underbrace{\int_a^b \frac{\sin x}{x}dx}_{I_1} + \underbrace{\int_a^b \frac{\cos x}{x^2}dx}_{I_2} \le \frac{2(b-a)}{a\cdot b} \qquad \text{(I)}$$

$$I_1 = \int_a^b \frac{1}{x}\sin x\, dx \quad \begin{pmatrix} u(x) = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}dx \\ dv = \sin dx \Rightarrow v(x) = -\cos x \end{pmatrix}$$

by using integration by part:

$$I_1 = \left[-\frac{1}{x}\cos x\right]_a^b - \int_a^b \frac{\cos x}{x^2}dx, I_1 + I_2 = \left[-\frac{1}{b}\cos(b) + \frac{1}{a}\cos(a)\right]$$

$$I_1 + I_2 = \frac{b\cos(a) - a\cdot\cos(b)}{a\cdot b} \quad (*)$$

Now let's prove: $b\cos a - a\cdot\cos(b) \overset{?}{\le} (b-a)\cdot 2$

$$b\cdot\cos a - 2b - a\cdot\cos b + 2a \overset{?}{\le} 0, \qquad b(\cos a - 2) + a(2 - \cos b) \overset{?}{\le} 0$$

$$b(\cos a - 2) \overset{?}{\le} a\cdot(\cos b - 2), \qquad \frac{\cos a - 2}{a} \overset{?}{\le} \frac{\cos b - 2}{b}$$

Suppose $f(x)=\frac{\cos x-2}{x}, x\in\,]0,\pi]$, $f'(x)=\frac{-x\sin x-\cos x+2}{x^2}=\frac{g(x)}{x^2}$

$$g(x)=-x\cdot\sin x-\cos x+2,\ 'g(x)=-(\sin x+x\cdot\cos x)-(\sin x)$$

$$'g(x)=-x\cdot\cos x\,,\ 'g(x)=0\Rightarrow x=\frac{\pi}{2}$$

x	0 $\qquad\qquad$ $\frac{\pi}{2}$ $\qquad\qquad$ π
$'g(x)$	$-\,-\,-\,-\,-\,-\,-\,-0+++++++$
$g(x)$	↘ $\frac{4-\pi}{2}$ ↗

So: $g(x)>0$

$'f(x)>0, f$ is completely increasing on $[0,\pi]$ so:$a\le b\Rightarrow f(a)\le f(b)$

$\frac{\cos a-2}{a}a\le\frac{\cos b-2}{b}$ and then: $b\cdot\cos a-a\cdot\cos(b)\le(b-a)$

and hence: $I_1+I_2\le\frac{2(b-a)}{a\cdot b}$