Prove that:

$$\left(\int_0^1 f(x)\,dx\right)^6 \le \left(\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{n+1}}\right)^3 \cdot \left(\int_0^1 f^2(x)\,dx\right)^2 \cdot \int_0^1 f^2(x)\varphi^3(x)\,dx,$$

where $f, \varphi : [0,1] \to (0,\infty)$ *are continous and* $\varphi(x) = \begin{cases} x^{x^2}, & x \in (0,1] \\ 1, & x = 0 \end{cases}$.

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Author solution: We note $A = \int_0^1 f^2(x)\varphi^3(x)\,dx, \quad B = \int_0^1 f^2(x)\,dx$ and according to $C - S - B$ inequality we have:

$$\left(\int_0^1 f(x)\,dx\right)^2 = \left(\int_0^1 f(x)\sqrt{2AB + B^2\varphi^3(x)} \cdot \frac{1}{\sqrt{2AB + B^2\varphi^3(x)}}\,dx\right)^2 \le$$

$$\le \int_0^1 (2AB + B^2\varphi^3(x)) \cdot f^2(x)\,dx \cdot \int_0^1 \frac{dx}{2AB + B^2\varphi^3(x)} \le$$

$$\le \left(2AB\int_0^1 f^2(x)\,dx + B^2\int_0^1 f^2(x)\cdot\varphi^3(x)\,dx\right) \cdot \int_0^1 \frac{dx}{3\sqrt[3]{A^2B^4}\,\varphi(x)} =$$

$$= \left(3AB^2\right) \cdot \frac{1}{3\sqrt[3]{A^2B^4}} \int_0^1 \frac{dx}{\varphi(x)} = \sqrt[3]{AB^2} \cdot \int_0^1 x^{-x^2}\,dx.$$

I'm calculating now:

$$\int_0^1 x^{-x^2}\,dx = \int_0^1 e^{-x^2 \ln x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}(\ln x)^n\,dx.$$

From the known integral:

$$I(m,n) = \int_0^1 x^m (\ln x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}},$$

it results that:

$$\int_0^1 x^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{n+1}},$$

where:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{n+1}} = 1 + \frac{1}{3^2} + \frac{1}{5^3} + ... = 1,11955...$$

Finally we have:

$$\left(\int_0^1 f(x)\, dx \right)^6 \leq \left(\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{n+1}} \right)^3 \cdot \left(\int_0^1 f^2(x)\, dx \right)^2 \cdot \int_0^1 f^2(x)\varphi^3(x)\, dx.$$