ROMANIAN MATHEMATICAL MAGAZINE

If $0 \leq x \leq 1, n \geq 1$ then:

$$\sum_{k=1}^{n}\int_{0}^{1}(cosx)^{k-1}\cdot(sinx)^{k}\,dx < 2\left(1-\frac{1}{2^n}\right)$$

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$$x \in [0,1] \subset \left[0,\frac{\pi}{2}\right) \Rightarrow sinx \geq 0, cosx > 0$$

$$(cosx)^{k-1}\cdot(sinx)^{k} = sinx\cdot(cosx)^{k-1}\cdot(sinx)^{k-1} =$$

$$= sinx\cdot(sinx\cdot cosx)^{k-1} = sinx\cdot(sin^2x\cdot cos^2x)^{\frac{k-1}{2}} \leq$$

$$\leq sinx\cdot\left(\left(\frac{sin^2x+cos^2x}{2}\right)^2\right)^{\frac{k-1}{2}} = sinx\cdot\left(\left(\frac{1}{2}\right)^2\right)^{\frac{k-1}{2}} = sinx\cdot\left(\frac{1}{2}\right)^{k-1} < \left(\frac{1}{2}\right)^{k-1}$$

$$\sum_{k=1}^{n}\int_{0}^{1}(cosx)^{k-1}\cdot(sinx)^{k}\,dx < \sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k-1} = \frac{\left(\frac{1}{2}\right)^n-1}{\frac{1}{2}-1} = 2\left(1-\frac{1}{2^n}\right)$$