ROMANIAN MATHEMATICAL MAGAZINE

If $\alpha > 2$, *then* :

$$2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx} < 2(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right)$$

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$$\text{Firstly}, x \geq \alpha > 2 \text{ and now}, \forall x \in (2,\infty), \frac{1-\cos x}{x^3} \overset{?}{<} \frac{2}{5x}$$

$$\Leftrightarrow 2x^2 + 5\cos x - 5 \underbrace{\overset{?}{>}}_{(*)} 0$$

$$\text{Let } F(x) = 2x^2 + 5\cos x - 5\ \forall x \in [2,\infty\rangle \text{ and then : } F'(x) = 4x - 5\sin x$$

$$\overset{\sin x \leq 1}{\geq} 4x - 5 \overset{x \geq 2}{\geq} 3 > 0 \Rightarrow F(x) \text{ is } \uparrow \text{ on } [2,\infty\rangle \Rightarrow F(x) \geq F(2) = 8 + 5\cos 2 - 5$$

$$\approx 0.919 > 0 \therefore \forall x \in (2,\infty), 2x^2 + 5\cos x - 5 > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \forall x \in (2,\infty), \frac{1-\cos x}{x^3} < \frac{2}{5x} \Rightarrow 2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx < 2\int_{\alpha}^{2\alpha}\frac{2}{5x}dx = \frac{4}{5}(\ln 2\alpha - \ln\alpha)$$

$$= \frac{4}{5}.\ln 2 \Rightarrow e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx} < e^{\left(\frac{4}{5}.\ln 2\right)} = \sqrt[5]{16} \therefore 2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx}$$

$$< 2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + \sqrt[5]{16} \overset{?}{<} 2(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right)$$

$$\Leftrightarrow \boxed{(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right) - \alpha.\cos\left(\frac{\pi}{\alpha}\right) \underbrace{\overset{?}{>}}_{(\bullet)} \frac{\sqrt[5]{16}}{2}}$$

$$\text{Let } f(t) = t.\cos\frac{\pi}{t} \text{ and then : } f'(t) = \frac{\pi}{t}.\sin\frac{\pi}{t} + \cos\frac{\pi}{t} \therefore \text{LHS of } (\bullet)$$

$$= f(\alpha+1) - f(\alpha) \overset{MVT}{=} ((\alpha+1)-\alpha)\frac{\pi}{\xi}.\sin\frac{\pi}{\xi} + \cos\frac{\pi}{\xi}\ (2 < \alpha < \xi < \alpha+1)$$

$$\therefore \text{LHS of } (\bullet) = \frac{\pi}{\xi}.\sin\frac{\pi}{\xi} + \cos\frac{\pi}{\xi} \to (1) \text{ and } \because \xi > 2 \therefore 0 < \frac{\pi}{\xi} < \frac{\pi}{2} \text{ and let}$$

$$P(\theta) = \theta.\sin\theta + \cos\theta\ \forall \theta \in \left[0,\frac{\pi}{2}\right) \therefore P'(\theta) = \theta.\cos\theta \geq 0 \Rightarrow P(x) \text{ is } \uparrow \text{ on } \left[0,\frac{\pi}{2}\right)$$

$$\Rightarrow P(x) \geq P(0) = 1 \therefore \forall \theta \in \left(0,\frac{\pi}{2}\right), \theta.\sin\theta + \cos\theta > 1 \therefore \text{via } (1), \text{LHS of } (\bullet) > 1$$

$$> \frac{\sqrt[5]{16}}{2} \Rightarrow (\bullet) \text{ is true} \therefore 2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx} < 2(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right)$$

$$\forall \alpha > 2\ (QED)$$