ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}\,dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ahmed Salem-Tunisia

$$\sum_{k=1}^{n}\cos^2 kx = \frac{1}{2}\left(\sum_{k=1}^{n}[2\cos^2(kx)-1+1]\right) = \frac{n}{2}+\sum_{k=1}^{n}\cos(2kx) =$$

$$= \frac{n}{2}\left(1+\frac{2}{n}\sum_{k=1}^{n}\cos(2kx)\right)$$

For $\alpha\geq 0, 1+\frac{1}{2}\alpha-\frac{1}{8}\alpha^2\leq\sqrt{1+\alpha}\leq 1+\frac{1}{2}\alpha$

$$\lim_{n\to\infty}\frac{1}{\sqrt{2}}\int_0^{2\pi}\left[1+\frac{1}{n}\sum_{k=1}^{n}\cos(2kx)-\frac{1}{2n^2}\left(\sum_{k=1}^{n}\cos^2(2kx)\right)\right]dx\leq\Omega\leq$$

$$\leq\lim_{n\to\infty}\frac{1}{\sqrt{2}}\int_0^{2\pi}\left(1+\frac{1}{n}\right)\sum_{k=1}^{n}\cos(2kx)\,dx$$

$$\int_0^{2\pi}\cos(2kx)\,dx = 0\ \forall k\in\mathbb{Z}_{\geq 1}$$

$$\int_0^{2\pi}\cos(2kx)\cos(2mx)\,dx = \begin{cases}\pi & if\ k=m\\ 0 & if\ k\neq m\end{cases}$$

$$\sqrt{2}\pi-\frac{1}{2\sqrt{2}}\lim_{n\to\infty}\frac{1}{n^2}n\pi\leq\Omega\leq\sqrt{2}\pi\Rightarrow\Omega=\sqrt{2}\pi$$

Solution 2 by Samir HajAli-Syria

$$\sum_{k=1}^{n}\cos^2(kx) = \sum_{k=1}^{n}\frac{1+\cos(k(2x))}{2} =$$

$$\sum_{k=1}^{n}\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{n}\cos(k(2x)) = \frac{1}{2}n+\frac{1}{2}\cdot\frac{\sin\left(\frac{n}{2}2x\right)\cdot\cos\left(\frac{n+1}{2}2x\right)}{\sin\left(\frac{2x}{2}\right)} =$$

$$\frac{1}{2}n+\frac{1}{2}\cdot\frac{\sin(nx)\cdot\cos(nx+k)}{\sin x}=\frac{1}{2}n+\frac{1}{2\sin x}\times\frac{1}{2}[\sin(2nx+x)+\sin(-x)]=$$

$$\frac{1}{2}n+\frac{1}{4\sin x}[\sin((2n+1)x)-\sin x]$$

$$\Rightarrow\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}=\frac{1}{2}+\frac{1}{4\sin x}\left[\frac{\sin((2n+1)x)}{n}-\frac{\sin x}{n}\right]$$

Let put: $f(n,x):=\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}$

Then:

$$\Omega=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}$$

$$\lim_{n\to+\infty}\int_0^{2\pi}\sqrt{\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}}=\int_0^{2\pi}\sqrt{\lim_{n\to+\infty}f(n,x)}\,dx=\int_0^{2\pi}\sqrt{\frac{1}{2}}dx=\frac{2\pi}{\sqrt{2}}=\pi\sqrt{2}$$

where: $\lim_{n\to+\infty}\frac{\sin x}{n}=0$ and

$$\lim_{n\to+\infty}-\frac{1}{n}\leq\lim\frac{\sin((2n+1)x)}{n}\leq\lim_{n\to+\infty}\frac{1}{n}\Rightarrow\lim_{n\to+\infty}\frac{\sin((2n+1)x)}{n}=0$$

So, $\lim_{n\to+\infty}f(n,x)=\frac{1}{2}+\frac{1}{4\sin x}(0-0)=\frac{1}{2}$

Solution 3 by Khaled Abd Imouti-Syria

$$\sum_{k=0}^{n}\cos^2(kx)=\frac{1}{4}\left(3+2n+\frac{\sin x\,(1+2n)}{\sin x}\right)$$

$$\frac{\sum_{k=0}^{n}\cos^2(kx)}{n}=\frac{\frac{3}{4}+\frac{1}{2}n+\frac{1}{4}\frac{\sin(x(1+2n))}{\sin x}}{n}$$

$$\frac{1+\sum_{k=1}^{n}\cos^2(kx)}{n}=\frac{3}{4n}+\frac{1}{2}+\frac{1}{4n}\cdot\frac{\sin(x(1+2n))}{\sin x}$$

$$\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}=-\frac{1}{4n}+\frac{1}{2}+\frac{1}{4n}\cdot\frac{\sin(x(1+2n))}{\sin x}$$

$$\lim_{n\to+\infty}\left(\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}\right)=\frac{1}{2},\qquad\Omega=\int_0^{2\pi}\frac{1}{\sqrt{2}}dx=\frac{1}{\sqrt{2}}\cdot 2\pi=\sqrt{2}\pi$$

Solution 4 by Amin Hajiyev-Azerbaijan

$$\Omega = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi} f_n^{\frac{1}{2}}(x)\,dx$$

$$f_n(x) = \sum_{k=1}^{n}\cos^2(kx) = \frac{1}{2}\sum_{k=1}^{n}(1+\cos(2kx)) = \frac{n}{2}+\frac{1}{2}\mathcal{R}\sum_{k=1}^{n} e^{2ikx} =$$

$$= \frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{\frac{e^{2ix}(e^{2inx}-1)}{e^{2ix}-1}\right\} = \frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{\frac{e^{2ix}\times e^{ikx}(e^{inx}-e^{-inx})}{e^{ix}(e^{ix}-e^{-ix})}\right\} =$$

$$= \frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{e^{ix(n+1)}\frac{\sin(nx)}{\sin(x)}\right\} = \frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{\left(\cos(x(n+1))+i\sin(x(n+1))\right)\frac{\sin(nx)}{\sin(x)}\right\} =$$

$$= \frac{n}{2}+\frac{1}{2}\cos(x(n+1))\frac{\sin(nx)}{\sin(x)} = \frac{n}{2}+\frac{\cos(xn)\sin(xn)\cos(x)}{2\sin(x)}-\frac{\sin^2(xn)}{2}$$

$$\Omega = \lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{f_n(x)}{n}}\,dx = \lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{1}{2}+\frac{\sin(2nx)}{4n}\tan(x)-\frac{\sin^2(nx)}{2n}}\,dx =$$

$$= \int_0^{2\pi}\sqrt{\lim_{n\to\infty}\left(\frac{1}{2}\right)+\frac{1}{4}\lim_{n\to\infty}\frac{\sin(2nx)}{n}\cot(x)-\frac{1}{2}\lim_{n\to\infty}\frac{\sin^2(nx)}{n}}\,dx\,;$$

$$0 < x < 2\pi\ \left\{n = \frac{1}{t};\, t\to 0\right\}$$

$$\Omega = \int_0^{2\pi}\sqrt{\frac{1}{2}+\frac{1}{4}\lim_{t\to 0} t\sin\left(\frac{2x}{t}\right)\cot(x)-\frac{1}{2}\lim_{t\to 0} t\sin^2\left(\frac{x}{t}\right)}\,dx = \int_0^{2\pi}\sqrt{\frac{1}{2}}\,dx = \pi\sqrt{2}$$

Answer:

$$\Omega = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}\,dx = \pi\sqrt{2}$$

Solution 5 by Adrian Popa-Romania

$$\sum_{k=1}^{n}\cos^2 kx = \sum_{k=1}^{n}\frac{1+\cos 2kx}{2} = \sum_{k=1}^{n}\frac{1}{2}+\sum_{k=1}^{n}\frac{\cos 2kx}{2} =$$

$$= \frac{n}{2}+\frac{1}{2}\cdot\frac{\sin nx\cdot\cos(n+1)x}{\sin x}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}\,dx=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{n}{2n}+\frac{1}{2}\cdot\frac{\sin nx\cdot\cos(n+1)x}{n\sin x}}\,dx$$

$$=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{1}{2}\left(1+\frac{\sin nx\cdot\cos(n+1)x}{n\sin x}\right)}$$

$$=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{1}{2}\left(1+\frac{\sin(2n+1)x-\sin x}{2n\sin x}\right)}\,dx$$

$$=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\int_0^{2\pi}\sqrt{1+\underbrace{\frac{\sin(2n+1)x}{n\sin x}}_{\to 0}-\underbrace{\frac{1}{2n}}_{\to 0}}\,dx=\frac{1}{\sqrt{2}}\int_0^{2\pi}dx=\frac{2\pi}{\sqrt{2}}=\sqrt{2}\pi$$