ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}} - \ln(n)\right), \qquad \omega_n = \lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k(x)}{\cos^2(x)}$$

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Solution by Shirvan Tahirov-Azerbaijan

$$\omega_n = \lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k(x)}{\cos^2(x)} = \lim_{x\to\frac{\pi}{2}} n!\frac{1-\sin^n(x)}{\cos^{2n}(x)} = n!\lim_{x\to\frac{\pi}{2}}\frac{1-\sin^n(x)}{\cos^{2n}(x)} \overbrace{\underbrace{=}_{k->0}}^{x=k+\frac{\pi}{2}}$$

$$= n!\lim_{k\to 0}\frac{-1-\cos^n(k)}{\sin^{2n}(k)} = n!\lim_{k\to 0}\frac{1-\cos^n(k)}{k^{2n}.\left(\frac{\sin^{2n}(k)}{k^{2n}}\right)} \overset{\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)}{\cong} n!\lim_{k\to 0}\frac{1-\cos^n(k)}{k^{2n}.\left(\frac{\sin(k)}{k}\right)^{2n}} =$$

$$= n!\lim_{k\to 0}\frac{\left(1-2\sin^2\left(\frac{k}{2}\right)\right)^{2n}-1}{k^{2n}} \overset{\lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)}{\cong} n!\lim_{k\to 0}\frac{2^n-(-1)^n(k^2-2)^n}{2^n.k^{2n}} = \frac{n!}{2^n}$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}} - \ln(n)\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\frac{k!}{2^k}}{2\,\frac{(k+1)!}{2^{k+1}}} - \ln(n)\right) =$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{2^{k+1}.k!}{2^{k+1}.(k+1)!} - \ln(n)\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{k!}{(k+1)} - \ln(n)\right) = \gamma - 1$$