ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\left(n\int_0^1 \frac{x^n\exp(x^2)}{1+x^{2n}}dx\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hikmat Mammadov-Azerbaijan

$$\Omega = \lim_{n\to\infty}\left(n\int_0^1 \frac{x^n\exp(x^2)}{1+x^{2n}}dx\right)$$

$$\int_0^1 \frac{x^n\exp(x^2)}{1+x^{2n}}dx = \int_0^1 x^n\exp(x^2)\sum_k(-1)^k x^{2nk}\,dx = \sum_{k=0}^{\infty}(-1)^k\int_0^1 \exp(x^2)\,x^{n(2k+1)}dx$$

$$\int_0^1 \exp(x^2)\,x^{n(2n+1)}\,dx =$$

$$= \frac{\exp(x^2)\,x^{n(2k+1)+1}}{n(2k+1)+1}\Bigg|_0^1 - \frac{2}{n(2k+1)+1}\int_0^1 \exp(x^2)\,x^{n(2k+1)+2}\,dx$$

$$= \frac{\exp(1)}{n(2k+1)+1} - \frac{2}{n(2k+1)+1}\int_0^1 \exp(x^2)\,x^{n(2k+1)+2}\,dx$$

$$\lim_{n\to\infty} n\sum_{k=0}^{\infty}(-1)^k\frac{\exp(1)}{n(2k+1)+1} = \lim_{n\to\infty}\sum_{k=0}^{\infty}(-1)^k\frac{\exp(1)\cdot n}{n(2k+1)+1} =$$

$$= \sum_{k=0}^{\infty}(-1)^k\lim_{n\to\infty}\frac{n\cdot\exp(1)}{n(2k+1)+1} =$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{\exp(1)}{2k+1} = \exp(1)\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \exp(1)\frac{\pi}{4} = e\frac{\pi}{4}$$

For another case: $\lim\limits_{n\to\infty} n\sum_{k=0}^{\infty}(-1)^k\frac{2}{n(2k+1)+1}\int_0^{12}\exp(x^2)\,x^{n(2k+1)+2}\,dx$

$$= \sum_{k=0}^{\infty}(-1)^k\lim_{n\to\infty}\frac{2n}{n(2k+1)+1}\cdot\lim_{n\to\infty}\exp(x^2)\,x^{m(2k\times1)+2} = 0$$

So we get:

$$\int_0^1 \frac{x^n\exp(x^2)}{1+x^{2n}}dx = e\frac{\pi}{4}$$

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Solution 2 by Exodo Halcalias-Angola

$$\lim_{n\to\infty}\left(n\int_0^1 \frac{x^n \exp(x^2)}{1+x^{2n}}dx\right)=\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1 \frac{y^{\frac{2k}{n}+\frac{1}{n}}}{1+y^2}dy\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k\in\mathbb{N}}\frac{1}{k!}\left(\sum_{j\in\mathbb{N}^*}(-1)^{j-1}\int_0^1 y^{2j-2+\frac{2k}{n}+\frac{1}{n}}dy\right)\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k\in\mathbb{N}}\frac{1}{k!}\left(\sum_{j\in\mathbb{N}^*}\frac{(-1)^{j-1}}{2j-1+\frac{2k}{n}+\frac{1}{n}}\right)\right)=\sum_{k\in\mathbb{N}}\frac{1}{k!}\left(\sum_{j\in\mathbb{N}^*}\frac{(-1)^{j-1}}{2j-1}\right)=$$

$$=e\int_0^1\frac{1}{1+y^2}dy=e\int_0^1 d\tan^{-1}(y)=\frac{\pi e}{4}$$

$$\therefore \lim_{n\to\infty}\left(n\int_0^1\frac{x^n\exp(x^2)}{1+x^{2n}}dx\right)=\frac{\pi e}{4}$$