ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\left(2^{-n}\cdot\prod_{k=1}^{n}\sqrt[k+1]{k}\right)$$

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$$\prod_{k=1}^{n}\sqrt[k+1]{k} = \sqrt{1}\cdot\sqrt[3]{2}\cdot\sqrt[4]{3}\cdot\ldots\cdot\sqrt[n+1]{n} =$$

$$= \sqrt{1\cdot 1}\cdot\sqrt[3]{1\cdot 1\cdot 2}\cdot\sqrt[4]{1\cdot 1\cdot 1\cdot 3}\cdot\ldots\cdot\sqrt[n+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{n\ times}\cdot n}\overset{MG\le MA}{<}$$

$$< \frac{2}{2}\cdot\frac{2\cdot 2}{3}\cdot\frac{2\cdot 3}{4}\cdot\ldots\cdot\frac{2\cdot n}{n+1}\Rightarrow$$

$$\Rightarrow 2^{-n}\prod_{k=1}^{n}\sqrt[k+1]{k} < \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{n}{n+1} = \frac{1}{n+1}\overset{n\to\infty}{\to} 0$$

$$\Rightarrow \lim_{n\to\infty} 2^{-n}\prod\sqrt[k+1]{k} = 0$$