ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\lim_{n\to 0}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^k(x)}dx\right)^k\right)$$

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Solution by Togrul Ehmedov-Azerbaijan

We know that

$$I_1=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^k(x)}dx=\int_0^{\frac{\pi}{2}}\frac{\cos^k(x)}{\sin^k(x)+\cos^k(x)}dx=\int_0^{\frac{\pi}{2}}\frac{\sin^k(x)}{\sin^k(x)+\cos^k(x)}dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin^k(x)+\cos^k(x)}{\sin^k(x)+\cos^k(x)}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{4}$$

$$I=\lim_{n\to 0}\left(\frac{1}{n}\sum_{k=1}^{n}(I_1)^k\right)=\lim_{n\to 0}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\pi}{4}\right)^k\right)=\lim_{n\to 0}\left(\frac{1}{n}\left(\frac{\frac{\pi}{4}\left(\left(\frac{\pi}{4}\right)^n-1\right)}{\frac{\pi}{4}-1}\right)\right)$$

$$=\frac{\pi}{\pi-4}\lim_{n\to 0}\left(\frac{\left(\frac{\pi}{4}\right)^n-1}{n}\right)=\frac{\pi}{\pi-4}\lim_{n\to 0}\left(\left(\frac{\pi}{4}\right)^n\log\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{\pi-4}\log\left(\frac{\pi}{4}\right)$$