ROMANIAN MATHEMATICAL MAGAZINE

If $\alpha,\beta,\gamma \geq 0$, then :

$$3+\frac{(\alpha+\beta+\gamma)^4}{135} \leq \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\sqrt[n^2]{e^{(k\alpha)^2}}\right)$$

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$$\text{Let } f(m)=e^m-1-m-\frac{m^2}{2}\ \forall\, m\geq 0 \text{ and then : } f'(m)=e^m-1-m\geq 0$$

$$\Rightarrow f(m) \text{ is } \uparrow \text{ on } [0,\infty) \Rightarrow f(m)\geq f(0)=1-1 \therefore e^m \geq 1+m+\frac{m^2}{2}\ \forall\, m\geq 0 \to (1)$$

$$\text{Now, } n^5-(n-1)^5=5n^4-10n^3+10n^2-5n+1 \text{ and putting}$$

$$n=1,2,3\ldots,(n-1),n \text{ successively, we arrive at :}$$

$$1^5-0^5=5.1^4-10.1^3+10.1^2-5.1+1$$

$$2^5-1^5=5.2^4-10.2^3+10.2^2-5.2+1$$

$$\ldots\ldots$$

$$(n-1)^5-(n-2)^5=5.(n-1)^4-10.(n-1)^3+10.(n-1)^2-5.(n-1)+1$$

$$n^5-(n-1)^5=5.n^4-10.n^3+10.n^2-5.n+1; \text{ and summing up, we arrive at :}$$

$$n^5=5\sum_{k=1}^{n}k^4-10\sum_{k=1}^{n}k^3+10\sum_{k=1}^{n}k^2-5\sum_{k=1}^{n}k+n$$

$$=5\sum_{k=1}^{n}k^4-10.\frac{n^2(n+1)^2}{4}+10.\frac{n(n+1)(2n+1)}{6}-5.\frac{n(n+1)}{2}+n$$

$$\Rightarrow \sum_{k=1}^{n}k^4=\frac{6n^5+15n^4+10n^3-n}{30}\to(2)$$

$$\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\sqrt[n^2]{e^{(k\alpha)^2}}\right)=\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}e^{\frac{k^2\alpha^2}{n^2}}\right)\overset{\text{via }(1)}{\geq}\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\left(1+\frac{k^2\alpha^2}{n^2}+\frac{k^4\alpha^4}{n^4}\right)\right)$$

$$=\sum_{k=1}^{n}\left(3+\frac{k^2}{n^2}\left(\sum_{cyc}\alpha^2\right)+\frac{k^4}{n^4}\left(\sum_{cyc}\alpha^4\right)\right)=3n+\frac{\sum_{cyc}\alpha^2}{n^2}.\sum_{k=1}^{n}k^2+\frac{\sum_{cyc}\alpha^4}{n^4}.\sum_{k=1}^{n}k^4$$

$$\overset{\text{via }(2)}{=}3n+\frac{\sum_{cyc}\alpha^2}{n^2}.\frac{n(n+1)(2n+1)}{6}+\frac{\sum_{cyc}\alpha^4}{n^4}.\frac{6n^5+15n^4+10n^3-n}{30}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\sqrt[n^2]{e^{(k\alpha)^2}}\right)$$

$$=\lim_{n\to\infty}\left(3+\frac{\sum_{cyc}\alpha^2}{6}.\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{\sum_{cyc}\alpha^4}{30}.\left(6+\frac{15}{n}+\frac{10}{n^2}-\frac{1}{n^4}\right)\right)$$

$$= 3+\frac{\sum_{cyc}\alpha^2}{3}+\frac{\sum_{cyc}\alpha^4}{5}\overset{\text{Holder}}{\geq}3+\frac{\sum_{cyc}\alpha^2}{3}+\frac{(\alpha+\beta+\gamma)^4}{5.27}\overset{\alpha,\beta,\gamma\geq 0}{\geq}3+\frac{(\alpha+\beta+\gamma)^4}{135}$$

$$\therefore 3+\frac{(\alpha+\beta+\gamma)^4}{135}\leq\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\sqrt[n^2]{e^{(k\alpha)^2}}\right)\ \forall\ \alpha,\beta,\gamma\geq 0\ (QED)$$