ROMANIAN MATHEMATICAL MAGAZINE

Let $x, y \geq 0$ such that $x + y = 1$.

Find the maximum and the minimum value of $P = \sin(\sqrt{x}) + \sin(\sqrt{y})$

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Since $t \to \sin t$ is increasing and concave on $[0, 1]$

then by Jensen's inequality, we have

$$P = \sin(\sqrt{x}) + \sin(\sqrt{y}) \leq 2\sin\left(\frac{\sqrt{x} + \sqrt{y}}{2}\right) \leq 2\sin\left(\sqrt{\frac{x+y}{2}}\right) = 2\sin\left(\frac{\sqrt{2}}{2}\right),$$

so the maximum value of P is $2\sin\left(\frac{\sqrt{2}}{2}\right)$, for $x = y = \frac{1}{2}$.

Also, by Jensen's inequality, we have

$$\sin(\sqrt{x}) \overset{\sqrt{x} \geq x}{\geq} \sin(x) = \sin(x.1 + y.0) \geq x \sin 1 + y \sin 0 = x \sin 1.$$

Similarly, we have $\sin(\sqrt{y}) \geq y \sin 1$.

Then $P \geq (x + y)\sin 1 = \sin 1$, so the minimum value of P is $\sin 1$,

for $x = 1$ and $y = 0$ or $x = 0$ and $y = 1$.