ROMANIAN MATHEMATICAL MAGAZINE

Let $x, y \in \mathbb{R}$. Find the maximum and the minimum value of

$$A = \sin x + \sin y + \sin(x+y)$$

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If M is the maximum value of A for $x = x_0$ and $y = y_0$ then we have

$$A = -[\sin(-x) + \sin(-y) + \sin(-x-y)] \geq -M, \quad \forall x, y \in \mathbb{R},$$

so $-M$ is the minimum value of P for $x = -x_0$ and $y = -y_0$.

Now, we have

$$A = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + \sin(x+y) = 2\left[\cos\left(\frac{x-y}{2}\right) + \cos\left(\frac{x+y}{2}\right)\right]\sin\left(\frac{x+y}{2}\right)$$

$$= 4\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\cos\left(\frac{z}{2}\right), \quad \text{where } z = \pi - (x+y).$$

If $\cos\left(\frac{x}{2}\right), \cos\left(\frac{y}{2}\right), \cos\left(\frac{z}{2}\right) \leq 0$, we have $A \leq 0$. WLOG we assume that $\cos\left(\frac{z}{2}\right) \geq 0$.

We have $A = 2\left[\cos\left(\frac{x-y}{2}\right) + \cos\left(\frac{x+y}{2}\right)\right]\cos\left(\frac{z}{2}\right) \leq 2\left[1 + \sin\left(\frac{z}{2}\right)\right]\cos\left(\frac{z}{2}\right)$

$$\overset{CBS}{\leq} 2\sqrt{(2+1)\left(\frac{1}{2} + \sin^2\left(\frac{z}{2}\right)\right)}\cos\left(\frac{z}{2}\right) \overset{AM-GM}{\leq} \sqrt{3}\left[\left(\frac{1}{2} + \sin^2\left(\frac{z}{2}\right)\right) + \cos^2\left(\frac{z}{2}\right)\right] = \frac{3\sqrt{3}}{2},$$

with equality for $\cos\left(\frac{x-y}{2}\right) = 1, \quad \sin^2\left(\frac{z}{2}\right) + \frac{1}{2} = \cos^2\left(\frac{z}{2}\right), \quad \sin\left(\frac{z}{2}\right) \geq 0 \Leftrightarrow$

$x, y \equiv \frac{\pi}{3}(mod 2\pi)$. So the maximum value of A is $\frac{3\sqrt{3}}{2}$ for $x, y \equiv \frac{\pi}{3}(mod 2\pi)$ and

the minimum value of A is $-\frac{3\sqrt{3}}{2}$ for $x, y \equiv -\frac{\pi}{3}(mod 2\pi)$.