ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \in [0, 2023]$. Find the max and min value of the expression :

$$P = a^3 + b^3 + c^3 - 2024abc$$

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By AM − GM inequality, we have

$$P \geq 3abc - 2024abc = -2021abc \geq -2021.2023^3,$$

then the min value of P is -2021.2023^3, reached at $a = b = c = 2023$.

Now, let $f(a) = P$, we have $f''(a) = 6a \geq 0$, then f is convex on $[0, 2023]$, and

$$P \leq max\{f(0), f(2023)\} = max\{b^3 + c^3, 2023^3 + b^3 + c^3 - 2024.2023bc\}.$$

Also, we have $b^3 + c^3 \leq 2.2023^3$, and if $g(b) = 2023^3 + b^3 + c^3 - 2024.2023bc$,

then g is convex on $[0, 2023]$, and $g(b) \leq max\{g(0), g(2023)\}$

$$= max\{2023^3 + c^3, 2.2023^3 + c^3 - 2024.2023^2c\} \overset{c \leq 2023}{\leq} 2.2023^3,$$

then the max value of P is 2.2023^3, reached at $a = b = 2023, c = 0$ and permutations.