ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{(a^4+b^4)h_c}{a+b}+\frac{(b^4+c^4)h_a}{b+c}+\frac{(c^4+a^4)h_b}{c+a}\ge 8\sqrt{3}\cdot F^2$$

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Solution 1 by Tapas Das-India

$$\frac{(a^4+b^4)h_c}{a+b}+\frac{(b^4+c^4)h_a}{b+c}+\frac{(c^4+a^4)h_b}{(c+a)}$$

$$\overset{AM-GM}{\ge}\frac{2a^2b^2\cdot 2F}{ac+bc}+\frac{2b^2c^2\cdot 2F}{ab+ac}+\frac{2c^2a^2\cdot 2F}{bc+ab}$$

$$=4F\left[\frac{a^2b^2}{ac+bc}+\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ab}\right]\ge 4F\frac{(ab+bc+ca)^2}{2(ab+bc+ca)}$$

$$=2F\,(ab+bc+ca)\ge 2F\ \times 4\sqrt{3}F=8\sqrt{3}F^2$$

Note: $ab+bc+ca\ge 4\sqrt{3}F$

$$ab+bc+ca\ge 3[(abc)^2]^{\frac{1}{3}}\ge 3\left(\frac{4F}{\sqrt{3}}\right)=4\sqrt{3}F$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{(a^2)^2+(b^2)^2}{\frac{a+b}{h_c}}\overset{CBS}{\ge}\frac{\left(2\sum a^2\right)^2}{2\sum\frac{a+b}{h_c}}=$$

$$=\frac{4\left(\sum a^2\right)^2}{\frac{2}{2F}\cdot\sum(a+b)c}=\frac{2F\left(\sum a^2\right)^2}{\sum ab}\ge\frac{2F\left(\sum a^2\right)^2}{\sum a^2}\ge 2F\cdot 4\sqrt{3}F=8\sqrt{3}F^2$$