POINT OPERATIONS

A NEW METHOD OF CREATING AND PROVING GEOMETRIC PROPOSITIONS

Nikos Iosifidis, Mathematician - Tutor, Veria

e-mail: iossifid@yahoo.gr

This paper (a part of it) was presented for the first time at the 31st Panhellenic Mathematics Conference organized in Veria on 7-11-14.
Here we present an improved and augmented version of it.

ABSTRACT - PURPOSE
In this paper a new concept **"POINT OPERATION"** is introduced **(abbreviated P.O).**
The purpose of this paper is to "algebraize" some proving procedures of Euclidean Geometry. We will present the first two chapters of the paper. We will rely only on two properties of P.O.

In the 1st chapter we show how we can create and at the same time prove in a very short way an infinite number of geometric propositions. For this purpose all we need to do is to write half a line at random and interpret it geometrically. Geometric interpretation is a very simple process. With minimal knowledge of Euclidean Geometry, anyone can create a geometric proposition without thinking, without difficulty, without auxiliary lines in a figure, in a minimum of time.
Propositions created by this process may be known but may be new. Some may be obvious, some may be easy to prove, but some may be difficult to prove with classical Euclidean Geometry.
In this chapter we show easy applications for understanding the process.

In the 2nd chapter we show how specific propositions are proved. The propositions we have chosen here are difficult or very difficult to prove with classical Euclidean Geometry. Using only the two properties of P.O the proofs of these difficult propositions do not exceed 2 lines.
In addition to the above, P.O have the following additional advantages:
Without any difficulty and in a short time they can generalize proven geometric propositions. This is done indicatively in examples 1, 15, 16, 20 and 21.
Also, from the proof of a proposition they can create new propositions related to the proven proposition in a minimum of time without any difficulty. This is done indicatively in examples 14 and 21.

Another key advantage of the method we describe is that all the proofs that follow are independent of the position of the points in the plane so that they apply in general and not only in the case of each figure.
Thus we can create various special cases (corollaries), which with classical Euclidean geometry would need a separate proof for each case of a different figure. This is done indicatively in examples 16 and 18.

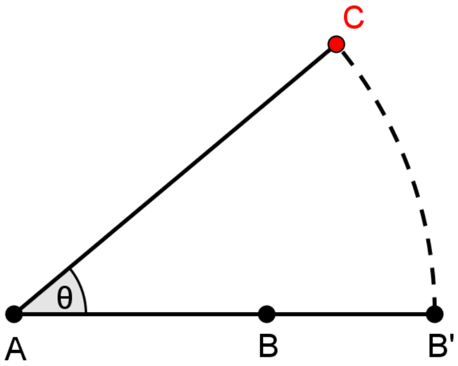
In the following, we will assume that all existing points belong in the same plane P and this will not be referred to in this paper.

For the congruence of two points A ≡ B we will use the simpler symbol A = B and we will read A equals B.

Angles will be considered as positively orientated and this will not be referred to in this paper.

We will use degrees as the unit of measurement of angles.

Definition of Point Operation (P.O)



Let $\lambda \in \mathbb{R}$ and $\theta \in [0^o, 360^o)$.

We name **Point Operation (λ, θ)** in P the mapping f: P X P⟶P with which the image of the pair (A, B) ∈ P X P is the point C of P which is defined as follows:
Let B′ be the homothetic of B with center A and ratio λ (λ> 0 or λ <0 or λ = 0), i.e. $\overrightarrow{AB'} = \lambda \cdot \overrightarrow{AB}$
Then point C is the rotation of B′ with center A and angle of rotation θ.
That is, a P.O. is a composition of two point transformations. A homothety and a rotation.
Such an operation will be denoted by o or *.
That is, we write AoB = C where o = (λ, θ).
Instead of saying that the image of the pair (A, B) is the point C, we also say more simply that the image of B with center A and P.O. the o is the point C.

It is obvious that:

- The image of the pair (A, B) is the same if the order of the transformations is reversed (i.e the rotation is applied first and then the homothety).

- If $\theta = 0$, the P.O $(\lambda, \theta) = (\lambda, 0)$ is the homothety with center A and ratio λ and the image AoB is the homothetic of B with center A and ratio λ, i.e. it is point B′.

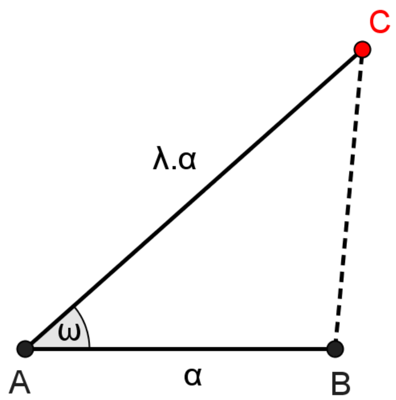
- If $\lambda = 1$, the P.O $(\lambda, \theta) = (1, \theta)$ is a rotation and the image AoB is the rotation of B around A at an angle θ.

That is, homothety and rotation are special cases of P.O.

- Obviously **AoA = A** is valid for every point A of P and for every P.O o.
- A special, but very common P.O is $o = (\frac{1}{2}, 0)$ where the image of the pairs (A, B) and (B, A) is the midpoint M of the segment AB.

  That is, **AoB = BoA = midpoint of the segment AB**

The truth of this equality is obvious.



- If two distinct points A and B are given, for each point C different from A there is a unique P.O o such that AoB = C

Indeed, if $\frac{AC}{AB} = \lambda$ and $\measuredangle BAC = \omega$ the required P.O is $o = (\lambda, \omega)$.

In the special case C = A, the P.O $o = (0, \omega)$ where ω any angle of the interval $[0^o, 360^o)$ has the property AoB = C.

Equivalently, if a triangle ABC is given then with P.O $o = (\lambda, \omega)$ where $\lambda = \frac{AC}{AB}$ and $\measuredangle BAC = \omega$ applies AoB = C

The following two basic properties apply:

If o = (λ, θ) and * = (μ, ω) are any two P.O and A, B, C, D any points, then

1) A * (B o C) = (A * B) o (A * C) and
(B o C) * A= (B * A) o (C * A) (I)

that is, any P.O is distributive in relation to any other P.O from both left and right.

2) (A o B) * (C o D) = (A * C) o (B * D) (I I)

The 2nd member of this equality arose by alternating the middle points B and C and also by alternating the operations o and *

In the above properties I and I I operation * is not precluded from being the same as operation o.

We point out that apart from two P.O $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 180)$ between two points A and B that give the midpoint of the segment AB, no other operation is commutative. We will not be concerned here with the proof of this proposition. The only thing we will need in this paper is that for the operation o = $(\frac{1}{2}, 0)$ the obvious applies:

AoB = BoA = midpoint of segment AB for each pair of points A and B.

Also the P.O are not associative, i.e. Ao(BoC) = (AoB)oC is not valid.
What is valid is the distributive property, i.e. Ao(BoC) = (AoB)o(AoC).

A geometric proof of the above properties I and I I would require the study of many cases (different combinations of λ and θ or different arrangement of points A, B, C, D or the case that some points are identical). For this reason we give the proof with the help of complex numbers which covers all possible cases. This is due to the fact that the proofs using the P.O apply to any arrangement of points in the plane, something we mentioned earlier.

To prove the properties I and I I we will rely on the following proposition:
Let o = (λ, θ) be a P.O and the corresponding complex $z = \lambda(\cos\theta + i\sin\theta)$
If for the points A, B, C AoB = C applies and a, b, c are the complexes with images the points A, B, C then the following applies: $c - a = (b - a)z$, therefore
$\mathbf{c = (b - a)z + a}$ (1)

Proof of property I I
We will prove that (AoB)*(CoD) = (A*C)o(B*D) (2)
where o = (λ, θ), * = (μ, ω) any two P.O and the complexes $z = \lambda(\cos\theta + i\sin\theta)$ and $w = \mu(\cos\omega + i\sin\omega)$ are the corresponding ones of the operations o and *.
Let AoB = E , CoD = F , A*C = G , B*D = H
and a, b, c, d, e, f, g, h are the complexes that have images at the points A, B, C, D, E, F, G, H respectively.
We will prove that E*F = GoH or K = L where K = E*F and L = GoH
If k, ℓ are the complexes with images the points K and L, it is enough to prove that $k = \ell$
According to (1) the following applies:

$e = (b - a)z + a \quad f = (d - c)z + c \quad g = (c - a)w + a \quad h = (d - b)w + b$ (3)

$$k = (f - e)w + e \overset{(3)}{=} (d + a - c - b)zw + (b - a)z + (c - a)w + a$$

$$\ell = (h - g)z + g \overset{(3)}{=} (d + a - c - b)zw + (b - a)z + (c - a)w + a \qquad (4)$$

From (3) and (4) it follows that k = ℓ

Proof of property I.
Because AoA = A for every point A and for every P.O o, according to the property I I will be:
A*(BoC) = (AoA)*(BoC) = (A*B)o(A*C) and
(BoC)*A = (BoC)*(AoA) = (B*A)o(C*A)

Simpler writing of properties I and I I

We simplify the above symbols of the operations so that they look more familiar and the properties of the operations are applied more easily.

When there is one operation we will call it "multiplication", we will symbolize it for convenience with · and we will read it as "by".

When there is another operation we will call it "addition", we will symbolize it with + and we will read it as "plus".

With these assumptions, symbolizing the operation o with + and the operation * with ·, the above properties can be written more simply as follows:

A·(B + C) = (A·B) + (A·C) and
(B + C)·A = (B·A) + (C·A) **(I)**

(A + B)·(C + D) = (A·C) + (B·D) **(I I)**

Assuming that "multiplication" precedes "addition" we can omit some parentheses and write the above properties even more simply as follows:

A·(B + C) = A·B + A·C and
(B + C)·A = B·A + C·A **(I)**

(A + B)·(C + D) = A·C + B·D **(I I)**

Finally, because the operation $\cdot = (\frac{1}{2}, 0)$ is very common, for reasons of simplification, we will write it as · = (δ, 0), i.e where the letter δ is present in an operation it will be understood as $\delta = \frac{1}{2}$

Continuing the presentation and if there are no more than two P.O we will use the last simplest writing of the properties I, I I.

CHAPTER 1

CREATION AND SIMULTANEOUS PROOF OF GEOMETRIC PROPOSITIONS

We will now use the properties I and I I as well as the commutativity of the operation (δ, 0) to create and prove geometric propositions.

As we said at the beginning of the paper, to create such propositions all we have to do is to write half a line at random and interpret it geometrically.
This half line is one of the properties I and I I written in exactly the same way or by replacing some points with some combination of other points.
Thus, the property I: $A\cdot(B + C) = A\cdot B + A\cdot C$ (1)
can be applied exactly as it is written and translating it geometrically with specific P.O + and · will give a specific geometric proposition. This proposition will apply to any arrangement of points A, B, C in the plane, whether points A, B, C are vertices of a triangle or are collinear.
By changing one or both operations, the same equality will give a different proposition that will again apply to any three points.
So for example if we choose as + = (δ, 0) and · = (1, 90) , (1) will give us a geometric proposition.
The same equality (1) if we choose other operations, eg + = (1, 60) and · = (2, 30) will give us a different proposition.
Thus, the relation (1) mentioned in 3 points can give infinite geometric propositions.
If for example A, B, C are vertices of an equilateral triangle it will give us a proposition related to the equilateral triangle. If A, B, C are vertices of a right triangle it will give a proposition related to the right triangle and so on.
In the same property that applies to any points A, B, C any of the points A, B, C may be the result of operations of other points. So this property can have, for example, the following forms (written at random):
$A\cdot(B + C\cdot D) = A\cdot B + A\cdot(C\cdot D)$
$(A\cdot B + B\cdot D)\cdot B = (A\cdot B)\cdot B + (B\cdot D)\cdot B$ where D is any other point.
Each of these equalities gives infinite new propositions for each choice of operations + and · and point D.

Property I I: $(A + B)\cdot(C + D) = A\cdot C + B\cdot D$ (2)
with the specific form and with specific P.O + and · gives a geometric proposition.
With another choice of operations + and · gives another proposition and so on, i.e. equality (2) can give infinite geometric propositions.
The same equality (2) applies of course if instead of points A, B, C, D we have some combinations of these points or combinations of other points.
Thus, for example (2) the following also applies (written at random):
$(A + B\cdot D)\cdot(B + (C + E)) = A\cdot B + (B\cdot D)\cdot(C + E)$ for any points A, B, C, D, E.

In this way it gives a new geometric proposition, i.e. the property I I gives infinite geometric propositions.
The propositions proved in this way (with properties I and I I) apply to every arrangement of the points in the plane and not only to the case of the respective figure. That is, property I I is valid whether the points A, B, C, D are vertices of a convex quadrilateral, or are vertices of a non-convex quadrilateral, or the 3 points or all 4 points are collinear, or some points are identical.

The operations that we will define and the equality that we will write for the creation or the proof of a proposition, can be related to a specific figure, so that a proposition related to that figure will emerge or they can be defined completely at random, so that some figure and a geometric proposition related to the defined operations and the specific figure will emerge. In this case the proposition that will be created will be revealed with the geometric interpretation of the equality we wrote.
In the following examples, sometimes we adapt the operations to a specific figure and sometimes we first define the operations and then design the figure that will result from the specific operations.
Either way the result is a known or new geometric proposition.

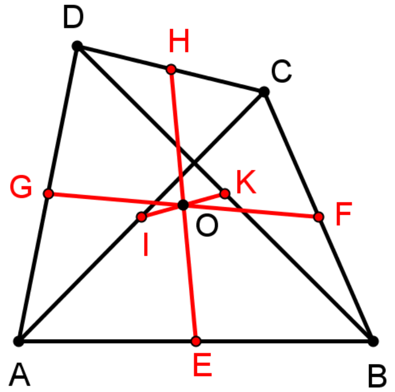
The above are understood from the following examples.

In the following, property I I will be named **"Fundamental Theorem of P.O" (Abbr: F.T)** because almost everything that follows is based on it.

Examples
1) For any 4 points A, B, C, D we define as multiplication and addition the same operation, $+ = (\delta, 0)$, $\cdot = (\delta, 0)$ and we apply the F.T.

The following applies: $\mathbf{(A + B)\cdot(D + C) = A\cdot D + B\cdot C}$ **(1)**

(The choice of operations is random. We do not know which proposition will emerge).
We now geometrically interpret the two members of (1)



1st member
A + B is the midpoint E of segment AB.
D + C is the midpoint H of segment DC.

2nd member
A·D is the midpoint G of segment AD.
B·C is the midpoint F of segment BC

The relation (1) is therefore written: E·H = G + F
E·H is the midpoint of the segment EH and G + F is the midpoint of GF.
We thus conclude that the segments EH and GF have a common midpoint.

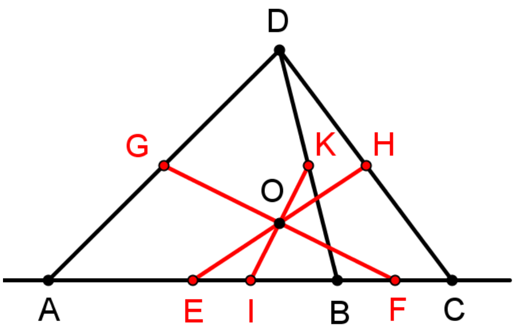
If I and K are the midpoints of the segments AC and BD and we take the relation **(A + B)·(C + D) = A·C + B·D** we conclude that the segments EH and IK have a common midpoint.

That is, we created and at the same time proved the following proposition:

If A, B, C, D are any 4 points and E, F, H, G, I, K the midpoints of the segments AB, BC, CD, DA, AC and BD respectively, then the segments EH, GF and IK have common midpoint.

If A, B, C, D are vertices of a convex quadrilateral the following proposition of Euclidean Geometry is well-known:

In each convex quadrilateral the segments connecting the midpoints of its opposite sides and the segment connecting the midpoints of its diagonals pass through the same point and are bisected.



Based on what we have explained earlier, it is not necessary that points A, B, C, D are vertices of a convex quadrilateral as with the above figure but can be any 4 points.

In the adjacent figure points A, B, C are collinear, but the proposition remains valid. The same proof applies of course in the case that all 4 points A, B, C, D are collinear.

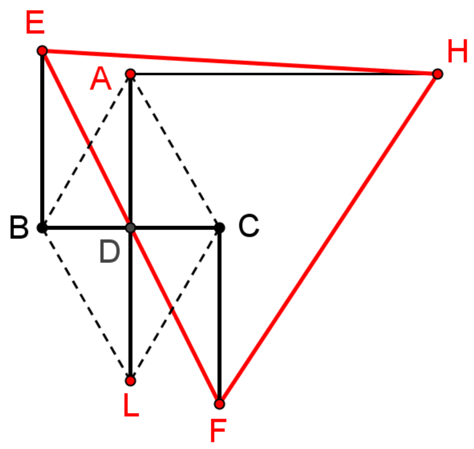
The generalization of the proposition is done without any difficulty in example 17, changing the operations from (δ, 0) and (δ, 0) to (λ, 0) and (μ, 0) and copying the same proof. Its generalization, i.e. example 17, is difficult to prove with classical Euclidean Geometry.

NOTE
For the reader's convenience we have noted in the text in red letters the points that arise from a P.O only the first time. Then, we write the same points in black. These points are also marked in red in the corresponding figures.

2) For the points B and C and P.O $+ = (1, 60)$ and $\cdot = (1, 90)$, we apply the F.T as follows:

(B + C)·(C + B) = B·C + C·B **(1)**

(The choice of operations is random. We do not know which proposition will arise). We now interpret relation (1) geometrically.



1st member

B + C is the rotation of point C around B at an angle of 60^o, i.e. is the vertex A of the equilateral triangle BCA of the figure and C + B is the vertex L of the equilateral triangle CBL of the figure.

The 1st member is now A·L. A·L is the point H which is the rotation of the point L around A at an angle of 90^o, i.e. AH ⊥ = AL and H = B·C + C·B applies.

2nd member

B·C is the rotation of C around the point B at an angle of 90^o, i.e. it is point E which is found if we draw the perpendicular to side BC at point B and we get on it a segment BE = BC.

C·B is point F of the figure which is found by drawing the perpendicular to BC at point C and get on it a segment CF = BC.

Therefore the previous relation is written H = E + F

The 2nd member of this equality is the third vertex of the equilateral triangle with side EF that resulted by rotating F around E at an angle of 60^o.

That is, we created and at the same time proved the following proposition:

Given an equilateral triangle ABC and the altitude AD we draw the perpendiculars to the side BC at the points B and C as in the figure and on them we get segments BE = CF = BC. We also draw the perpendicular to AD at point A and on it we get segment AH = 2AD as in the figure. Then the triangle EFH is equilateral.

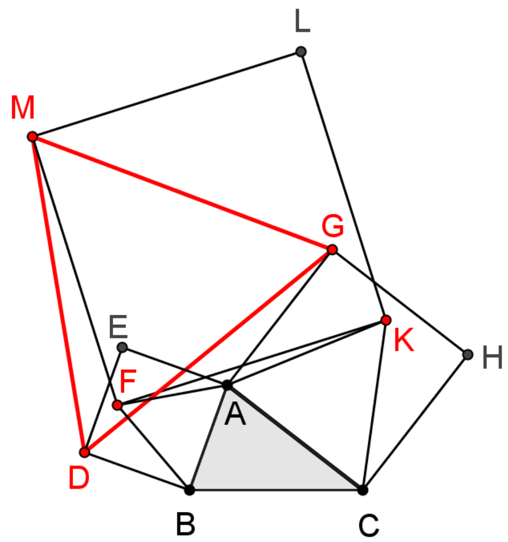
3) Let ABC be any triangle and the P.O $+ = (1, 90)$ and $\cdot = (1, 60)$

(Random operations)

The following applies: **(B + A)·(A + C) = B·A + A·C** **(1)**

We now geometrically interpret relation (1).

1st member
B + A is the vertex D of the square BAED of the figure, and A + C is the vertex G of the square ACHG of the figure.

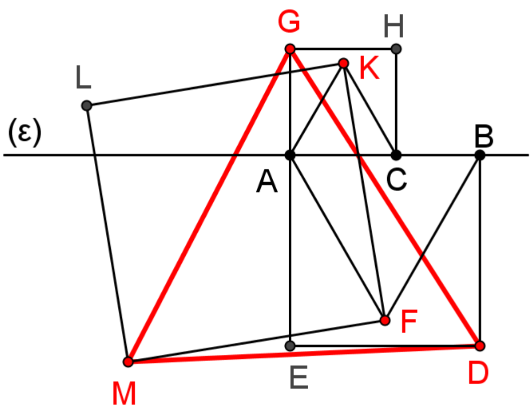


2nd member
B·A is the vertex F of the equilateral triangle BAF of the figure and A·C is the vertex K of the equilateral triangle ACK of the figure.
The 2nd member of (1) is now F + K which is the vertex M of the square FKLM of the figure.

Therefore (1) becomes: D·G = M
The last relation means that M is the third vertex of the equilateral triangle of the figure with side DG.

That is, we created and at the same time proved the following proposition:

Exterior to a triangle ABC we construct the squares ABDE, ACHG and the equilateral triangles ABF and ACK of the figure. We also construct the square FKLM of the figure. Then the triangle DGM is equilateral.



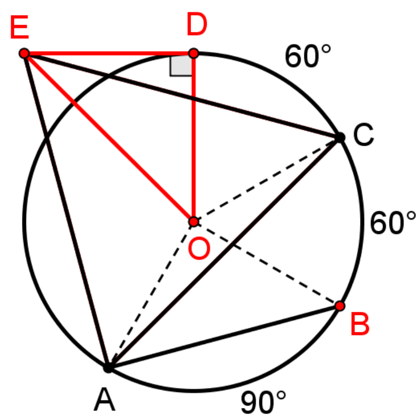
Again A, B, C do not have to be triangle vertices. They can also be collinear. It also does not matter how they are arranged in a straight line. As long as the squares and the equilateral triangles have the same orientation just as the example we just showed. In the adjacent figure the points A, B, C are collinear but the proposition continues to be valid.

The arrangement of points A, B, C on the same line does not matter either. With a different layout, e.g. A between B and C, the figure changes, but the proposition is valid again.
With classical Euclidean Geometry we will be obliged to make a separate proof for each of these cases.

4) In a circle O we consider the successive arcs $AB = 90^o$ and $BC = 60^o$
We define the operations: + = (1, 90) and · = (1, 60)
With these operations the following applies: O + A = B and B·C = O
(Here we have adapted the operations to the figure. We are waiting for a proposition related to the figure).
We apply the F.T related to the data.

(O + A)·C = O·C + A·C **(1)**

We geometrically interpret the two members of (1).



1st member
The 1st member of this relation is the point B·C = O

2nd member
O·C is the point D of the circle, such that $\measuredangle COD = 60^o$

A·C is the vertex E of the equilateral triangle ACE of the figure.

Thus the relation (1) is now written as:
O = D + E
This means that DO ⊥ = DE, so ED is tangent to the circle and equal to its radius.

That is, we created and at the same time proved the following proposition:

In a circle of radius R the successive arcs $AB = 90^o$, $BC = 60^o$, $CD = 60^o$ are provided. We construct the equilateral triangle ACE of the figure. Then ED is tangent to the circle and ED = R.

5) Consider the triangle ABC and the P.O + = (δ, 0) and · = (1, 90)
For a random point X the following applies: (B + X)·A = B·A + X·A.
We consider A·C as X. That is, we have

(B + A·C)·A = B·A + (A·C)·A **(1)**

(Random selection of operations and point X. We do not know which proposition will be revealed).
We now geometrically interpret the two members of (1)

1st member

$A \cdot C$ is the vertex H of the square ACFH of the figure.

$B + A \cdot C = B + H$ is the midpoint I of the segment BH.

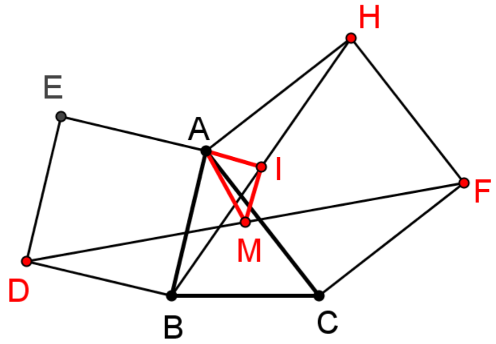
The 1st member of (1) is equal to $I \cdot A$

2nd member

$B \cdot A$ is the vertex D of the square ABDE of the figure.

$(A \cdot C) \cdot A = H \cdot A$ is point F.

So, the 2nd member is point $D + F$ which is the midpoint M of the segment DF.



Thus relation (1) is written $I \cdot A = M$

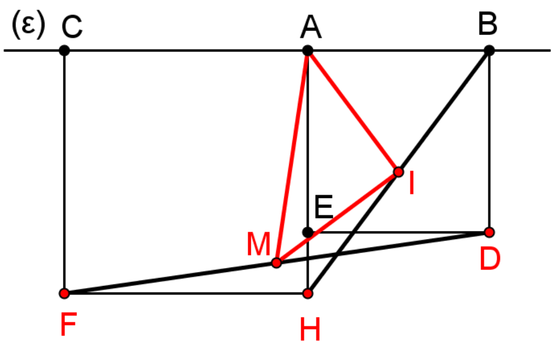
So M is the rotation of A around I at an angle of 90^{o}, i.e. $IM \perp = IA$ which means that the triangle IAM is right-angled at I and isosceles.

That is, we created and at the same time proved the following proposition:

A triangle ABC is given and the squares ABDE and ACFH exterior to the triangle. If M and I are the midpoints of DF and BH respectively then the triangle IAM is right-angled at I and isosceles.

As we explained, A , B , C do not have to be triangle vertices. They can also be collinear, as long as the squares in both cases have the same orientation.

In the figure below the points A , B , C are collinear, but the proposition remains valid.



6) Let ABC be any triangle. We construct the equilateral triangles ABD and ACE of the figure. We consider the operation $\cdot = (1, 60)$ related to the figure.

With this operation $B \cdot A = D$ and $A \cdot C = E$ apply.

We also consider the operation $+ = (\delta, 0)$ and apply the relation:

$$(B + A) \cdot (A + C) = B \cdot A + A \cdot C \qquad (1)$$

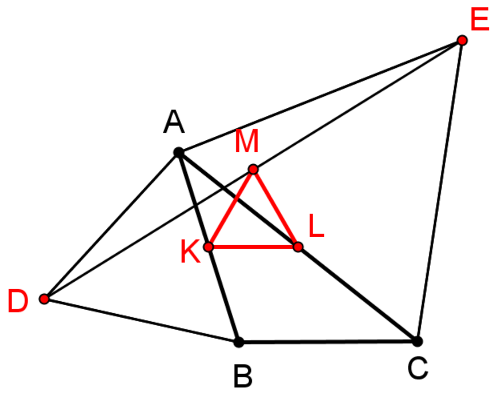
We now geometrically interpret the two members of (1)

1st member

B + A is the midpoint K of AB and A + C is the midpoint L of AC.
Thus the 1st member is equal to K·L

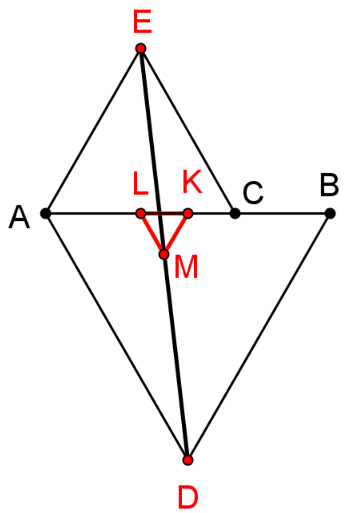
2nd member
The following apply: B·A = D and
A·C = E
So the 2nd member of (1) is equal to
D + E which is the midpoint M of
segment DE.

Relation (1) is now written as K·L = M
K·L is the rotation of L around K at an angle of 60°, i.e. it is the third vertex of the equilateral triangle with side KL.



That is, we created and at the same time proved the following proposition:

ABC triangle is given. Exterior to the triangle we construct the equilateral triangles ABD and ACE. If K, L and M are the midpoints of AB, AC and DE respectively, then the triangle KLM is equilateral.



The same proof is valid if points A, B, C are collinear with any arrangement, as long as the equilateral triangles in both cases have the same orientation.
In the adjacent figure, the same proposition is presented with points A, B, C being collinear.

7) Given a triangle ABC and the squares ABDE and ACFH exterior to it.
We consider the P.O · = (1, 90) relative to the figure.
According to this operation the following apply: B·A = D and A·C = H
We also consider the P.O + = (δ, 0)
Apply F.T for points A, B, C as follows:

(B + A)·(A + C) = B·A + A·C **(1)**

We now interpret (1) geometrically.

1st member
B + A is the midpoint K of AB and A + C is the midpoint L of AC.
1st member of (1) is now K·L

2nd member

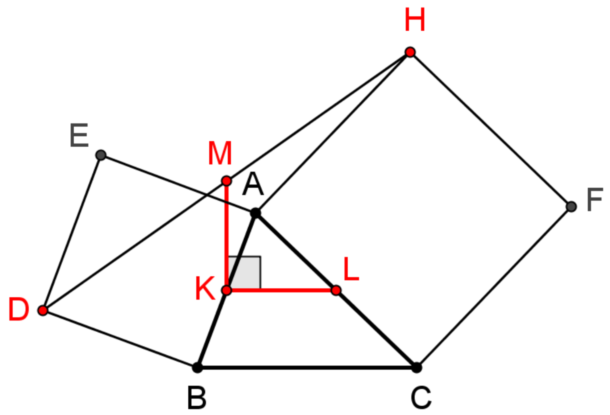
B·A = D and A·C = H apply

The 2nd member of (1) is therefore D + H which is the midpoint M of the segment DH.

(1) is now written K·L = M

K·L is the rotation of L around K at an angle of 90^o which means that

$KM \perp = KL$



Because $KL // = \dfrac{BC}{2}$, we created and at the same time proved the following proposition:

ABC triangle is given. Construct the squares ABDE and ACFH exterior to the triangle. If K and M are the midpoints of the segments AB, and DH respectively, then MK is perpendicular to the side BC and equal to half of it.

The same proof applies if points A, B, C are collinear with any arrangement, as long as the squares in both cases have the same orientation.

8a) In the circle O of the figure two arcs $AB = CD = 60^o$ are given. We define an operation $\cdot = (1, 60)$ such that A·B = O and C·D = O

(The operation · related to the figure).

Apply the F.T as follows:

(A·B)·(C·D) = (A·C)·(B·D) **(1)**

We geometrically interpret (1)

1st member

1st member is O·O = O

2nd member

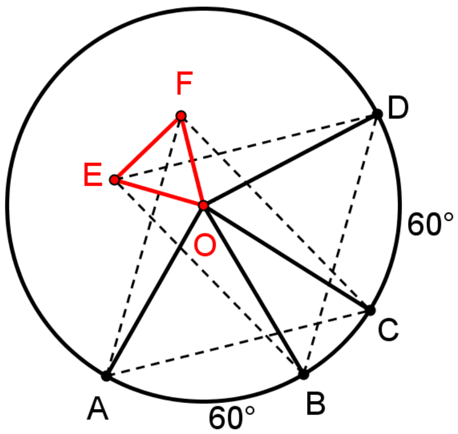
A·C is the vertex F of the equilateral triangle ACF of the figure.

B·D is the vertex E of the equilateral triangle BDE of the figure.

Thus, relation (1) is written: O = F·E.

F·E is the rotation of E around F at an angle 60^o.

This means that the triangle OFE is



equilateral.

That is, we created and at the same time proved the following proposition:

In a circle O we consider the points A, B, C, D so that $AB = CD = 60^o$ and the arcs have the same direction. Construct the equilateral triangles ACF and BDE of the figure. Then the triangle OEF is equilateral.

NOTE

The proof shows that the relative position of the points A, B, C, D can be any with the only restriction that the arcs AB and CD must have the same direction for A·B = O and C·D = O to be valid.

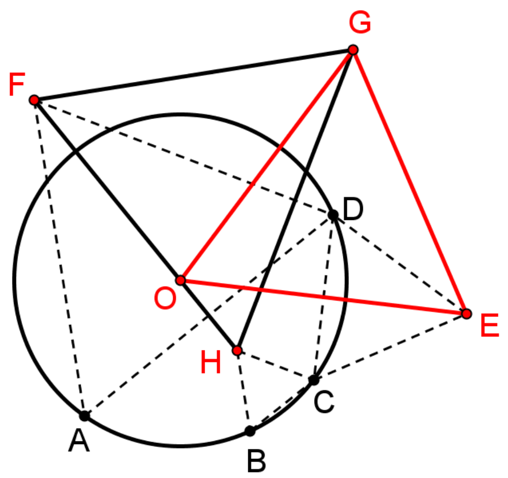
8b) In the same figure and with the same operation · = (1, 60) we now use the relation

(A·D)·(B·C) = (A·B)·(D·C) **(1)**

We geometrically interpret the two members of (1)

1st member

A·D is the vertex F of the equilateral triangle ADF of the figure.
B·C is the vertex H of the equilateral triangle BCH of the figure.
So the 1st member of (1) is the point F·H which is the third vertex G of the equilateral triangle FHG.



2nd member

A·B is the point O
D·C is the third vertex E of the equilateral triangle DCE.

Thus relation (1) is written G = O·E
The last equality means that the triangle OEG is also equilateral.

That is, we created and at the same time proved the following proposition:

In circle O are given the points A, B, C, D so that the arcs AB and CD have the same orientation and $AB = CD = 60^o$.
We construct the equilateral triangles ADF, BCH, DCE and FHG of the figure.

Then the triangle OEG is equilateral.

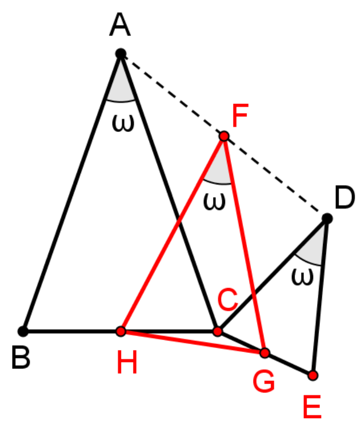
9) Consider two similar and isosceles triangles ABC (AB = AC) , DCE (DC = DE) and let ∡A = ∡D = ω as in the figure.

We define an operation · = (1, ω) so that A·B = C and D·C = E.
We define another operation + = (δ, 0) and let us consider the relation

(A + D)·(B + C) = A·B + D·C **(1)**

We interpret the above relation geometrically.

1st member
A + D is the midpoint F of the segment AD.
B + C is the midpoint H of the segment BC.



2nd member
It is equal to C + E as we explained and is the midpoint G of CE.
Thus relation (1) is written: F·H = G
F·H is the rotation of H around F at an angle ω, i.e
FH = FG and ∡HFG = ω

Therefore the triangle FHG is isosceles and is similar to ABC.

That is, we created and at the same time proved the following proposition:

In the above figure the isosceles triangles ABC (AB = AC) and DCE (DC = DE) are similar. If F, H, G are the midpoints of AD, BC and CE respectively then the triangle FHG is similar to ABC.

With the previous examples we showed how someone with minimal knowledge can create and at the same time prove a geometric proposition. The examples we gave were quite simple in order to easily understand this process.
We also can make much more complex and difficult geometric propositions using the properties I and I I in a more complex way. We give such an example.

10) Use more than two P.O
Consider any 4 points A, B, C, D and the P.O o = (δ, 0), * = (1, 60) and ● = (1, 90)
Based on property I, the following applies:
Ao((B*C)●D) = (Ao(B*C))●(AoD) = ((AoB)*(AoC))●(AoD), i.e

Ao((B*C)●D) = ((AoB)*(AoC))●(AoD) **(1)**

We interpret its two members geometrically (1)

1st member

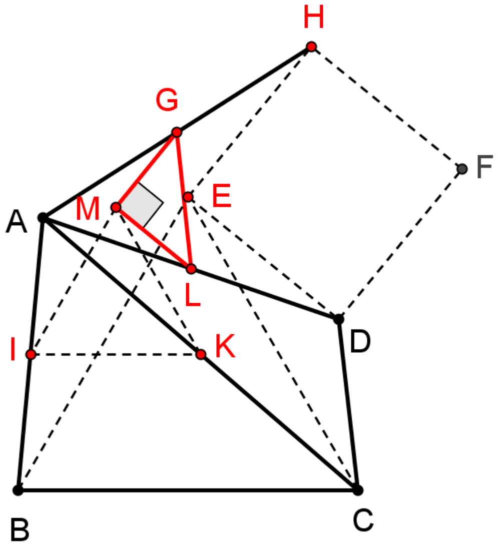
B*C is the vertex E of the equilateral triangle BCE of the figure. Thus its 1st member of (1) is Ao(E●D)

E●D is the vertex H of the square EDFH.

The 1st member of (1) is now AoH.

AoH is the midpoint G of the segment AH.

Finally, the 1st member of (1) is the midpoint G of the segment AH.



2nd member

AoB is the midpoint I of segment AB

AoC is the midpoint K of segment AC

AoD is the midpoint L of segment AD

The 2nd member of (1) is therefore the point (I*K)●L

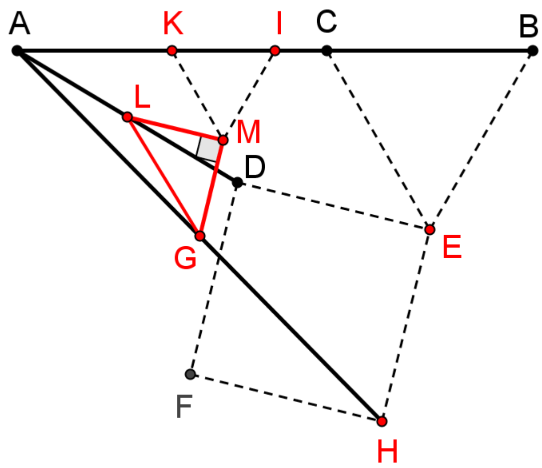
I*K is the vertex M of the equilateral triangle IKM

So the 2nd member of (1) is now the point M●L

Thus, the relation (1) becomes G = M●L

The point M●L is the rotation of L around M at an angle of 90^o.

This means that the segments ML and MG are equal and perpendicular or in other words, the triangle MLG is right-angled at M and isosceles.



So we created and at the same time proved the following proposition:

Let A, B, C, D be any points and I, K the midpoints of AB, AC respectively. We construct the equilateral triangles BCE and IKM and the square EDFH of the figure. Let G, L be the midpoints of AH, AD respectively. Then the triangle MGL is right-angled at M and isosceles.

As we have explained, the proposition we created is valid to any arrangement of points A, B, C, D. It also applies in the event that some points are congruent.

In the figure below the points A, B, C are collinear, but the proposition is valid again.

CHAPTER 2
PROOFS OF GEOMETRIC PROPOSITIONS

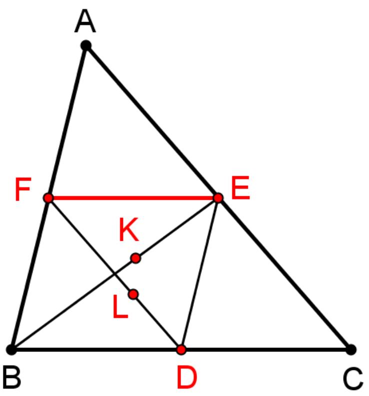
The following examples show how we can prove specific geometric propositions.

11) The segment that connects the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.

Proof

Let F and E be the midpoints of the sides AB and AC of triangle ABC respectively.

We will prove that $FE // = \frac{BC}{2}$



Let D be the midpoint of the side BC. We will prove that the quadrilateral BDEF is a parallelogram. Suffice it to show that the diagonals of the quadrilateral BDEF are bisected, so

FE // = BD, i.e. $FE // = \frac{BC}{2}$

We consider the P.O $+ = (\delta, 0)$

The following applies to the midpoints K and L of BE and DF:

K = B + E and L = D + F

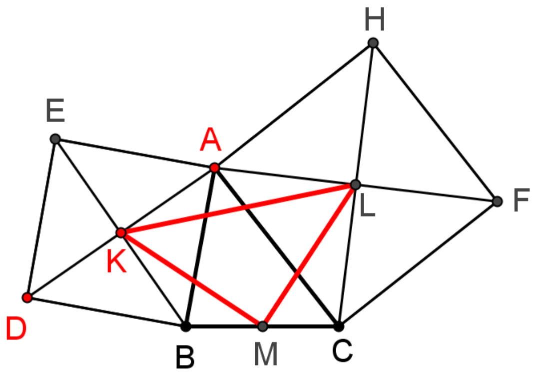
We will prove that B + E = D + F (1)

For midpoints D, E, F of BC, CA, AB the following apply:

D = B + C , E = C + A , F = B + A

So (1) is written B + (C + A) = (B + C) + (B + A) that is valid (property I)

12) A triangle ABC and the squares of the figure ABDE and ACFH are given. If K and L are the centers of these squares and M is the midpoint of BC prove that the triangle MLK is right-angled and isosceles.



Proof

Consider the operations $+ = (\delta, 0)$ and $\cdot = (1, 90)$

We will prove that M·L = K

The following applies:

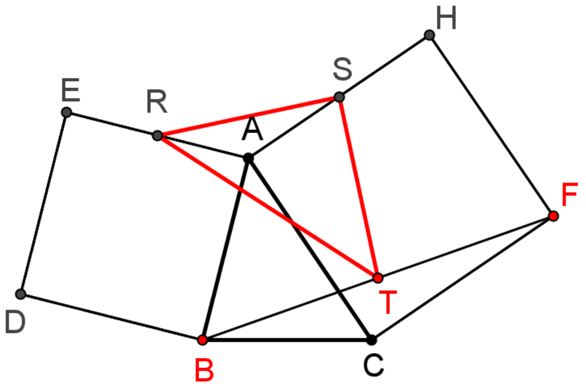
M·L = (B + C)·(A + F) = B·A + C·F = D + A = K

13) Exterior to the triangle ABC we construct the squares ABDE and ACFH. If R, S, T are the midpoints of AE, AH and BF respectively, prove that the triangle RST is right-angled and isosceles.

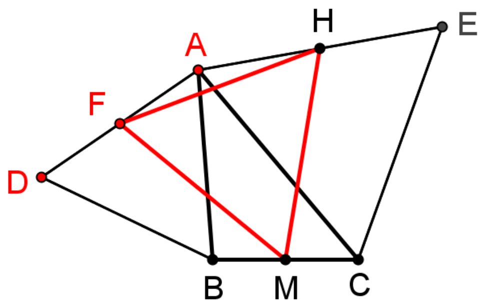
Proof

Consider the operations + = (δ, 0)
and · = (1, 90)
We will show that S·R = T

The following applies:
S·R = (H + A)·(A + E) =
H·A + A·E = F + B = T



14) Exterior to the triangle ABC we construct the equilateral triangles ABD and ACE. If F, H, M are the midpoints of AD, AE and BC respectively, show that the triangle MFH is equilateral.



Proof

Consider the operations: + = (δ, 0)
and · = (1, 60)
We will prove that M·H = F
The following applies: M·H = (B + C)·(A + E) = B·A + C·E = D + A = F

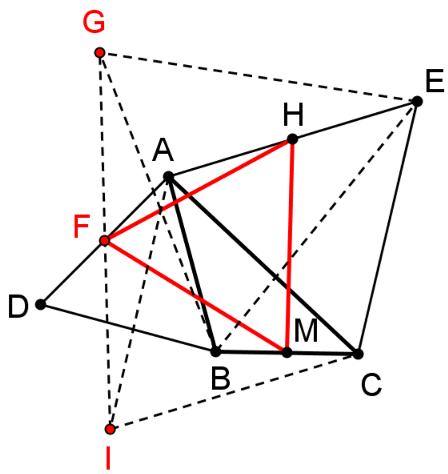
Creating a new relevant proposition

If in the relation M·H = F we write M = B + C and H = E + A then another proposition emerges (Fig. 2).

Indeed, the relation M·H = F ⇒

(B + C)·(E + A) = F ⇒ B·E + C·A = F

B·E is the vertex G of the equilateral triangle BEG of the figure and C·A is the vertex I of the equilateral triangle CAI of the figure.
Thus the previous relation is written:
G + I = F which means that F is the midpoint of the segment GI.

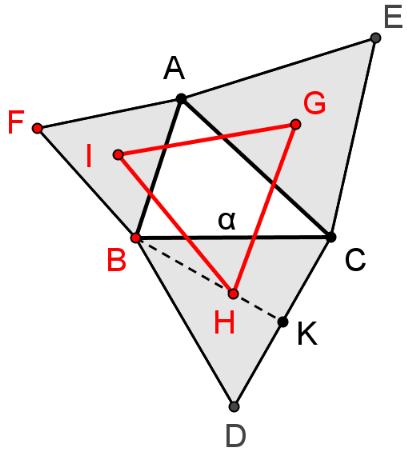


15) Napoleon's theorem
Exterior to the triangle ABC we construct the equilateral triangles BCD, CAE and ABF. If H, G, I are their respective centers, prove that the triangle HGI is equilateral.

Proof

If BC = α, then $BH = \frac{2}{3}BK = \frac{2}{3}\cdot\frac{\alpha\sqrt{3}}{2} = \frac{\alpha\sqrt{3}}{3}$

$\Rightarrow \frac{BH}{BD} = \frac{\sqrt{3}}{3}$



Consider the operations + = (λ, 30) with $\lambda = \frac{\sqrt{3}}{3}$

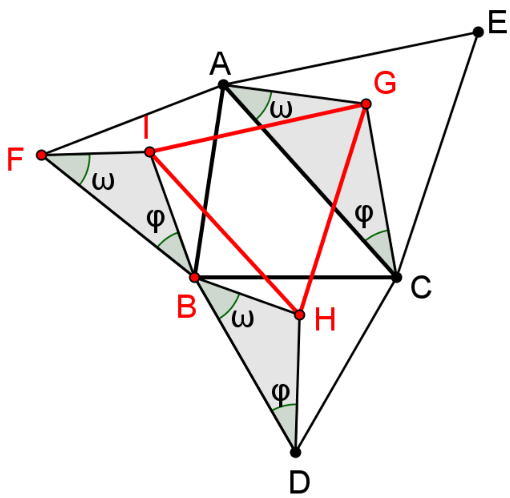
and · = (1, 60)
Therefore
B + D = H, A + C = G, F + B = I (1)
We will prove that H·G = I
Indeed,
H·G = (B + D)·(A + C) = B·A + D·C = F + B = I

Generalization of the theorem
The proof shows that H, G, I do not have to be the centers of the equilateral triangles BCD, CAE, ABF, as long as the relations (1) are valid with any other P.O.
This can be done with a P.O + = (λ, ω) as follows:
We construct the similar triangles BDH, ACG and FBI of the figure with

$\frac{BH}{BD} = \frac{AG}{AC} = \frac{FI}{FB} = \lambda$ and

$\measuredangle DBH = \measuredangle CAG = \measuredangle BFI = \omega$,

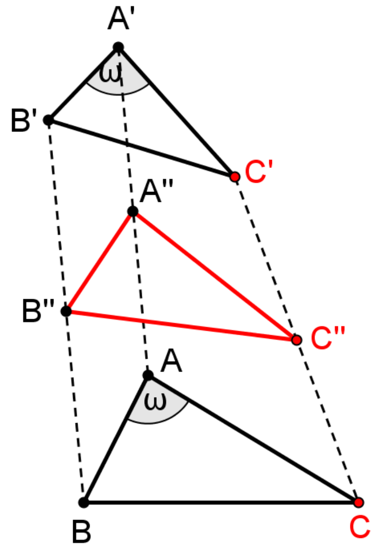
so the relations (1) are valid and the triangle HGI is equilateral.



16) The triangles ABC and A′B′C′ are similar with A = A′, B = B′, C = C′ and with the same orientation.
If A″, B″, C″ are the midpoints of AA′, BB′, CC′ respectively, prove that the triangle A″B″C″ is similar to the triangle ABC.

Proof:

Let $\measuredangle BAC = \measuredangle B'A'C' = \omega$ and $\dfrac{AC}{AB} = \dfrac{A'C'}{A'B'} = \lambda$

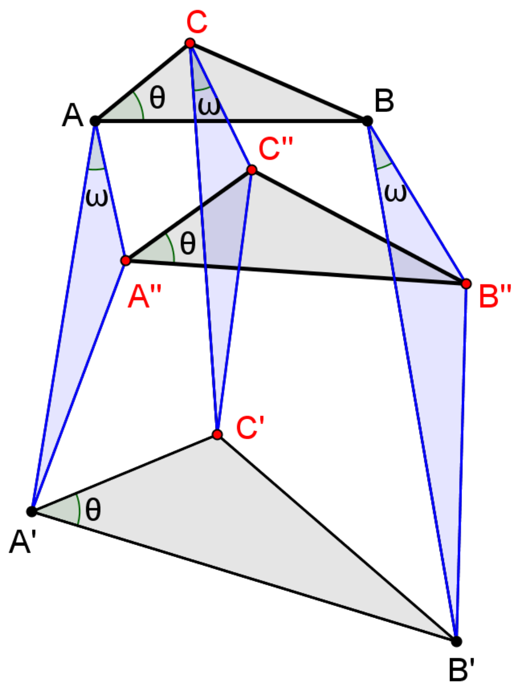


Consider the operations: $+ = (\delta, 0)$ and
$\cdot = (\lambda, \omega)$
We will show that $A''\cdot B'' = C''$
The following applies: $A''\cdot B'' = (A + A')\cdot(B + B')$
$= A\cdot B + A'\cdot B' = C + C' = C''$
and the proposition has been proved.

Generalization of the proposition
It follows from the proof that the points A″, B″, C″ need not be the midpoints of AA′, BB′, CC′, as long as they divide these segments into equal ratios, i.e.
$\dfrac{\overrightarrow{A''A}}{\overrightarrow{A''A'}} = \dfrac{\overrightarrow{B''B}}{\overrightarrow{B''B'}} = \dfrac{\overrightarrow{C''C}}{\overrightarrow{C''C'}}$, so the following will also apply $\dfrac{\overrightarrow{AA''}}{\overrightarrow{AA'}} = \dfrac{\overrightarrow{BB''}}{\overrightarrow{BB'}} = \dfrac{\overrightarrow{CC''}}{\overrightarrow{CC'}} = \mu$
As a P.O + we then consider the P.O $+ = (\mu, 0)$.

The above proof is valid of course even if some points are congruent.

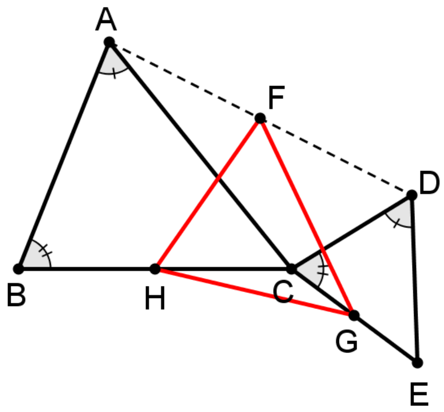
Further generalization (Theorem of Petersen – Schoute)
If as an operation + we consider
$+ = (\mu, \omega)$ where $\mu = \dfrac{AA''}{AA'}$ we have the following even more general proposition:



Let be two similar triangles ABC and A′B′C′ of the same orientation with homologous vertices (A, A′), (B, B′), (C, C').
With sides AA′, BB′, CC′ we construct the similar triangles AA′A″, BB′B″, CC′C″ of the same orientation with homologous vertices (A, B, C), (A′, B′, C′**), (A″, B″, C″).**
Then the triangle A″B″C″ is similar to ABC.

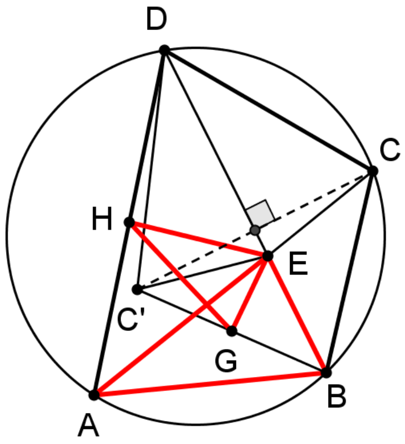
Creating special propositions (corollaries)
With the help of the above general proposition we can create more specific propositions such as the following (i) and (ii).

i) In the following figure the triangles ABC and DCE are similar with homologous vertices (A, D), (B, C) and (C, E).
If F, H, G are the midpoints of the segments AD, BC and CE respectively, then that the triangle FHG is similar to ABC.



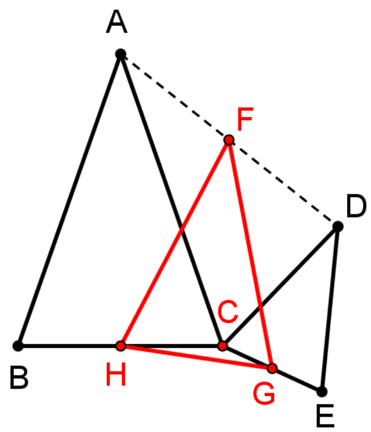
This is a special case of the previous general proposition, because the triangles ABC and DCE are similar with homologous vertices (A, D), (B, C) and (C, E) and the vertices F, H, G of the triangle FHG are the midpoints of AD, BC and CE respectively.

ii) In a circle there is inscribed a quadrilateral ABCD whose diagonals intersect at E. Let C′ be the symmetric of C with respect to the diagonal BD and H and G the midpoints of the segments AD and BC′ respectively.
Then the triangles EAB and EGH are similar.



Indeed, the triangles EAB and ECD are similar. The triangle EC′D is congruent to ECD. In the similar triangles EAB and EDC′ the homologous vertices are (E, E), (A, D) and (B, C') and the vertices E, H, G of the triangle EHG are the midpoints of the segments EE, AD and BC′ respectively. Thus the EHG triangle is similar to EAB.

A special case of the above proposition is example 9 that we created in the 1st part of this paper. Recall the figure.



17) USEFUL PROPOSITION (Generalization of example 1)
A convex quadrilateral ABCD and the points E, F, H, G of sides AB, BC, CD and DA respectively are given, such that: $\mathbf{\frac{AE}{AB}=\frac{DH}{DC}=\lambda}$ **and** $\mathbf{\frac{AG}{AD}=\frac{BF}{BC}=\mu}$

If $\mathbf{S = EH \cap GF}$**, prove that** $\mathbf{\frac{GS}{GF}=\lambda}$ **and** $\mathbf{\frac{ES}{EH}=\mu}$

Because $\frac{AE}{AB}=\frac{DH}{DC} \Leftrightarrow \frac{AE}{AB-AE}=\frac{DH}{DC-DH} \Leftrightarrow \frac{EA}{EB}=\frac{HD}{HC}=\lambda_1$

the proposition can be formulated equivalently as follows:

If $\frac{EA}{EB}=\frac{HD}{HC}=\lambda_1$ and $\frac{GA}{GD}=\frac{FB}{FC}=\mu_1$ then $\frac{SG}{SF}=\lambda_1$ and $\frac{SE}{SH}=\mu_1$

With the second writing the proposition is easier to memorize.

Proof

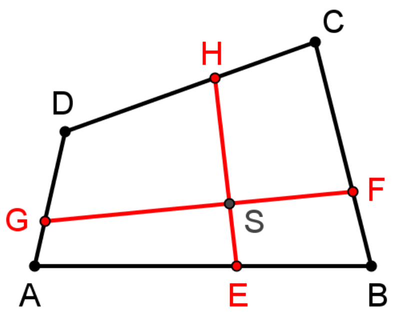
Let P.O + = (λ, 0) and · = (μ, 0).

With these operations the following apply:

E = A + B , H = D + C , F = B·C , G = A·D (1)

We will prove that S = G + F and S = E·H i.e. E·H = G + F or, because of relations (1) it suffices to prove that:

(A + B)·(D + C) = A·D + B·C which is valid (F.T)



Proving the above proposition with classical Euclidean Geometry is difficult.

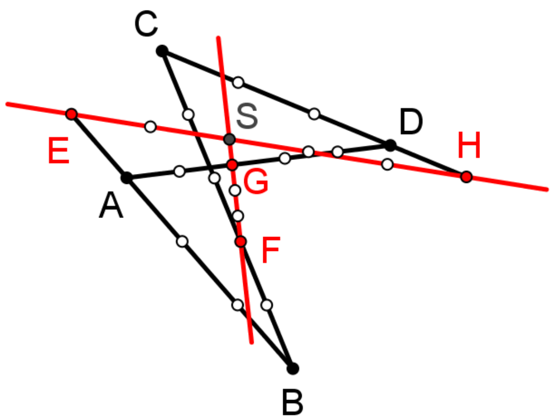
Generalization

The proof shows that the proposition does not only apply to points on the sides of a convex quadrilateral, but the exact same proof applies in general as follows:

4 points A, B, C, D are given and the points E, F, H, G of the lines AB, BC, CD and DA respectively, so that

$\frac{\overrightarrow{AE}}{\overrightarrow{AB}}=\frac{\overrightarrow{DH}}{\overrightarrow{DC}}=\lambda$ **and**

$\frac{\overrightarrow{AG}}{\overrightarrow{AD}}=\frac{\overrightarrow{BF}}{\overrightarrow{BC}}=\mu$



If the lines EH and GF intersect at S, then $\frac{\overrightarrow{GS}}{\overrightarrow{GF}}=\lambda$ **and** $\frac{\overrightarrow{ES}}{\overrightarrow{EH}}=\mu$

Equivalently: If $\frac{\overrightarrow{EA}}{\overrightarrow{EB}}=\frac{\overrightarrow{HD}}{\overrightarrow{HC}}=\lambda_1$

and $\frac{\overrightarrow{GA}}{\overrightarrow{GD}}=\frac{\overrightarrow{FB}}{\overrightarrow{FC}}=\mu_1$ **and the lines EH and GF intersect at S, then**

$\frac{\overrightarrow{SG}}{\overrightarrow{SF}}=\lambda_1$ **and** $\frac{\overrightarrow{SE}}{\overrightarrow{SH}}=\mu_1$

Eg in the following figure is $\dfrac{\overrightarrow{EA}}{\overrightarrow{EB}} = \dfrac{\overrightarrow{HD}}{\overrightarrow{HC}} = \dfrac{1}{4}$ and $\dfrac{\overrightarrow{GA}}{\overrightarrow{GD}} = \dfrac{\overrightarrow{FB}}{\overrightarrow{FC}} = -\dfrac{2}{3}$

The lines EH and GF intersect at S and $\dfrac{\overrightarrow{SG}}{\overrightarrow{SF}} = \dfrac{1}{4}$ and $\dfrac{\overrightarrow{SE}}{\overrightarrow{SH}} = -\dfrac{2}{3}$ apply.

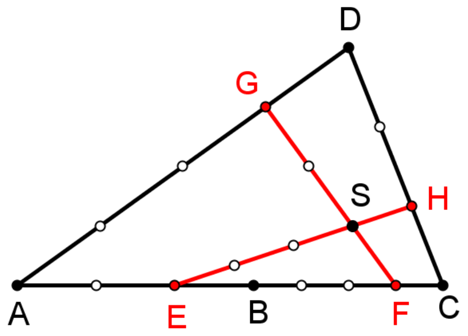
NOTE

To memorize the proposition in the general case, we design the points A, B, C, D as vertices of a convex quadrilateral ABCD, we consider the points of the divisions on the sides of ABCD and write the relation for the convex quadrilateral as we did before the generalization. Then, the same relation will apply to any arrangement of points A, B, C, D and points of divisions.

In the next figure, points A, B, C are collinear and the proposition continues to hold.

In this figure are $\dfrac{\overrightarrow{EA}}{\overrightarrow{EB}} = \dfrac{\overrightarrow{HD}}{\overrightarrow{HC}} = -2 = \lambda$ and $\dfrac{\overrightarrow{GA}}{\overrightarrow{GD}} = \dfrac{\overrightarrow{FB}}{\overrightarrow{FC}} = -3 = \mu$

The lines EH and FG intersect at S and holds $\dfrac{\overrightarrow{SG}}{\overrightarrow{SF}} = \lambda = -2$ and $\dfrac{\overrightarrow{SE}}{\overrightarrow{SH}} = \mu = -3$



18) A triangle ABC is given, the points D, F of the side AB and the points E, H of the side AC such that $\frac{BD}{BA}=\frac{AE}{AC}=\lambda$ and $\frac{BF}{BA}=\frac{AH}{AC}=\mu$.

If S is the intersection point of DE and FH prove that $\frac{DS}{DE}=\mu$ and $\frac{FS}{FH}=\lambda$

The proposition can be formulated equivalently as follows:

If $\frac{DB}{DA}=\frac{EA}{EC}=\lambda_1$ and $\frac{FB}{FA}=\frac{HA}{HC}=\mu_1$ then $\frac{SD}{SE}=\mu_1$ and $\frac{SF}{SH}=\lambda_1$

With the second form, the proposition is easier to remember.

Proof

Let P.O + = (λ, 0) and · = (μ, 0).

We will prove that S = D·E και S = F + H

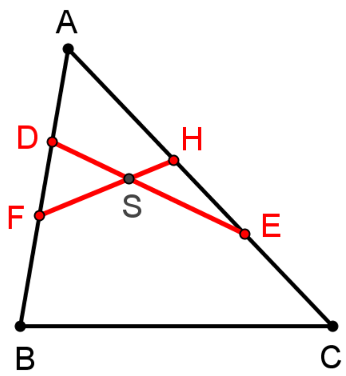
Suffice it to show that D·E = F + H (1)

According to the above operations the following apply:

D = B + A, E = A + C, F = B·A and H = A·C

So (1) is written: (B + A)·(A + C) = B·A + A·C which is valid (F.T)

We emphasize that the proposition is valid even if the points D, E, F, H are not points of the sides of the triangle, but points of the lines of its sides, because in the P.O (λ, θ) it can be λ> 1 or λ <0. The property I and the F.T are valid for each P.O.

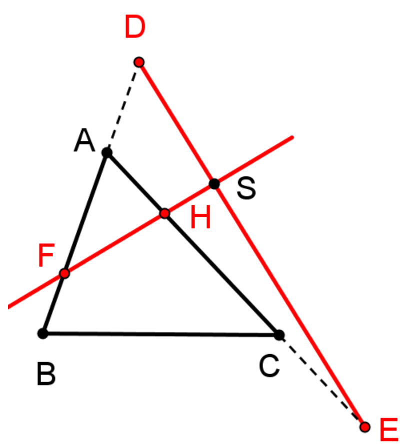


In the general case the proposition applies as follows:

If $\frac{\overrightarrow{BD}}{\overrightarrow{BA}}=\frac{\overrightarrow{AE}}{\overrightarrow{AC}}=\lambda$ and $\frac{\overrightarrow{BF}}{\overrightarrow{BA}}=\frac{\overrightarrow{AH}}{\overrightarrow{AC}}=\mu$ then $\frac{\overrightarrow{DS}}{\overrightarrow{DE}}=\mu$ and $\frac{\overrightarrow{FS}}{\overrightarrow{FH}}=\lambda$

Equivalently:

If $\frac{\overrightarrow{DB}}{\overrightarrow{DA}}=\frac{\overrightarrow{EA}}{\overrightarrow{EC}}=\lambda_1$ and $\frac{\overrightarrow{FB}}{\overrightarrow{FA}}=\frac{\overrightarrow{HA}}{\overrightarrow{HC}}=\mu_1$ then $\frac{\overrightarrow{SD}}{\overrightarrow{SE}}=\mu_1$ and $\frac{\overrightarrow{SF}}{\overrightarrow{SH}}=\lambda_1$

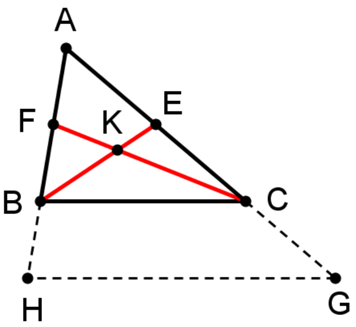


The previous proof also applies to the general case.

For example in the adjacent figure it is: $\frac{\overrightarrow{DB}}{\overrightarrow{DA}} = \frac{\overrightarrow{EA}}{\overrightarrow{EC}} = 3 = \lambda$ and $\frac{\overrightarrow{FB}}{\overrightarrow{FA}} = \frac{\overrightarrow{HA}}{\overrightarrow{HC}} = -\frac{1}{2} = \mu$

The lines DE and FH intersect at S and is valid again

$\frac{\overrightarrow{SD}}{\overrightarrow{SE}} = -\frac{1}{2} = \mu$ and $\frac{\overrightarrow{SF}}{\overrightarrow{SH}} = 3 = \lambda$



The above proposition is useful in many cases.

As an application of this proposition we prove that:

The medians of each triangle intersect at the same point at a distance from each vertex $\frac{2}{3}$ of the corresponding median.

Proof

Let ABC be any triangle and the medians BE and CF intersect at point K.
In the extensions of AB and AC we get the points H and G respectively, such that BH = BF and CG = CE.
Then in the triangle AHG with the division points F and B on the side AH and the corresponding points C and E on the side AG, we have:

$\frac{FH}{FA} = \frac{CA}{CG} = 2 = \lambda$ and $\frac{BH}{BA} = \frac{EA}{EG} = \frac{1}{2} = \mu$

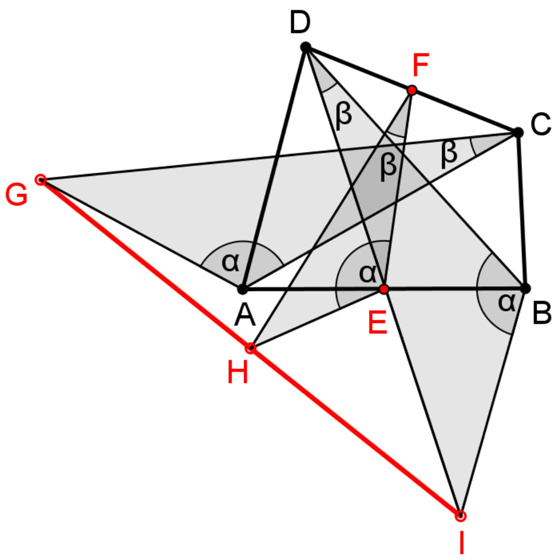
According to the above proposition, it is $\frac{KF}{KC} = \mu = \frac{1}{2}$ and $\frac{KB}{KE} = \lambda = 2$

Therefore $BK = \frac{2}{3}BE$ and $CK = \frac{2}{3}CF$

19) A quadrilateral ABCD (convex or non-convex) and the midpoints E and F of the sides AB and CD respectively are given.
We construct the similar triangles EFH, ACG, BDI of the same orientation with

$\measuredangle E = \measuredangle A = \measuredangle B$, $\measuredangle F = \measuredangle C = \measuredangle D$.

Prove that H is the midpoint of GI.
(This proposition is the reverse of proposition 16)



Proof

Let $\frac{EH}{EF} = \frac{AG}{AC} = \frac{BI}{BD} = \lambda$ and

$\measuredangle FEH = \measuredangle CAG = \measuredangle DBI = \alpha$

Consider the P.O

$+ = (\delta, 0)$ and $\cdot = (\lambda, \alpha)$

Then the following apply: $E = A + B$, $F = C + D$, $H = E{\cdot}F$, $G = A{\cdot}C$, $I = B{\cdot}D$

To prove that H is the midpoint of GI, it suffices to show that $G + I = H$

Indeed: $G + I = A{\cdot}C + B{\cdot}D =$

$(A + B){\cdot}(C + D) = E{\cdot}F = H$

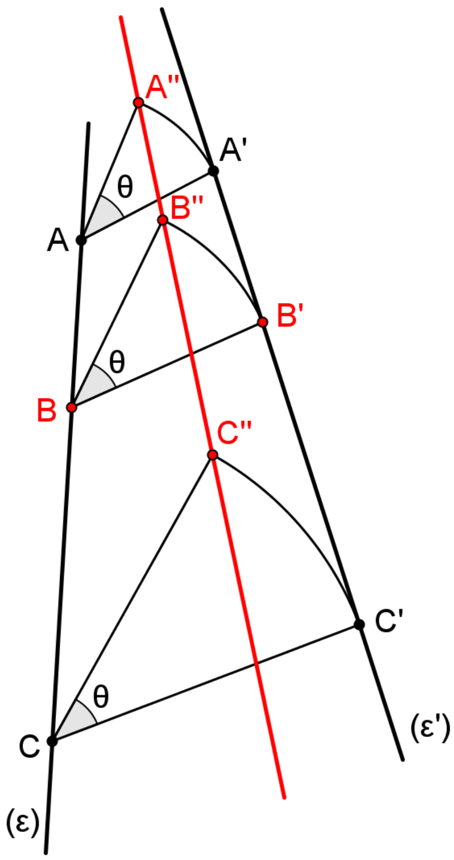
So H is the midpoint of GI.

20) The points A, B, C in this order belong to a line (ε) and the points A', B', C' in this order belong to another line (ε'). Apply $\frac{AB}{AC} = \frac{A'B'}{A'C'} = \lambda$

We rotate the points A', B', C' around A, B, C respectively at an angle θ.

Let A″, B″, C″ be the corresponding images.

Prove that the points A″, B″, C″ are collinear and $\frac{A''B''}{A''C''} = \lambda$



Proof

Consider the P.O $+ = (\lambda, 0)$ and $\cdot = (1, \theta)$

Then the following applies:

$A + C = B$, $A' + C' = B'$, $A{\cdot}A' = A''$,

$B{\cdot}B' = B''$, $C{\cdot}C' = C''$

We have to prove that $B'' = A'' + C''$

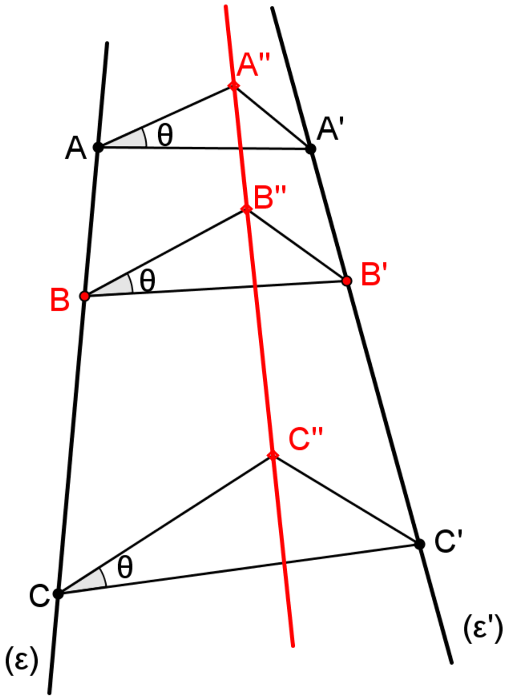
Indeed, $B'' = B{\cdot}B' = (A + C){\cdot}(A' + C') =$

$A{\cdot}A' + C{\cdot}C' = A'' + C''$

The last equality proves what is required.

Generalization

If A, B, C are collinear, as well as A', B', C' and $\dfrac{\overrightarrow{AB}}{\overrightarrow{AC}} = \dfrac{\overrightarrow{A'B'}}{\overrightarrow{A'C'}} = \lambda$ and the triangles $AA'A''$, $BB'B''$, $CC'C''$ are similar with homologous vertices (A, A', A''), (B, B', B'') και (C, C', C'') and have the same orientation, then A'', B'', C'' are collinear and $\dfrac{\overrightarrow{A''B''}}{\overrightarrow{A''C''}} = \lambda$



Proof

Let $\dfrac{AA''}{AA'} = \dfrac{BB''}{BB'} = \dfrac{CC''}{CC'} = \mu$ and

$\measuredangle A = \measuredangle B = \measuredangle C = \theta$

It suffices now to consider as a P.O the $\cdot = (\mu, \theta)$ and to copy the previous proof.

21) In a circle O we consider the arcs $AB = CD = EF = 60^o$ of the figure. Let K, L, M be the midpoints of the chords FA, BC and DE respectively. Prove that the triangle KLM is equilateral.

Proof

Consider the P.O $+ = (\delta, 0)$ and $\cdot = (1, 60)$

We will show that $K{\cdot}L = M$ (Fig. 1)

Indeed: $K{\cdot}L = (F + A){\cdot}(C + B) =$

$\underline{F{\cdot}C + A{\cdot}B} = F{\cdot}C + O = F{\cdot}C + O{\cdot}O =$

$(F + O){\cdot}(C + O) = \underline{(F + O){\cdot}(O + C)} =$

$F{\cdot}O + O{\cdot}C = E + D = M$

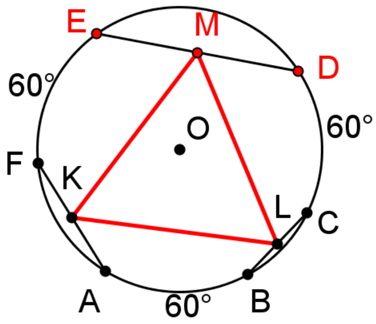


Fig. 1

The proof shows that the points A, B, C, D, E, F do not have to be consecutive, as long as the arcs AB, CD and EF are directed the same way.

Generalization of the proposition.
Without any change, the proof of the general proposition is made, according to which the equilateral triangles OAB, OCD and OEF are not congruent, as long as the angles $\measuredangle AOB$, $\measuredangle COD$ and $\measuredangle EOF$ are directed positively (Fig. 2).

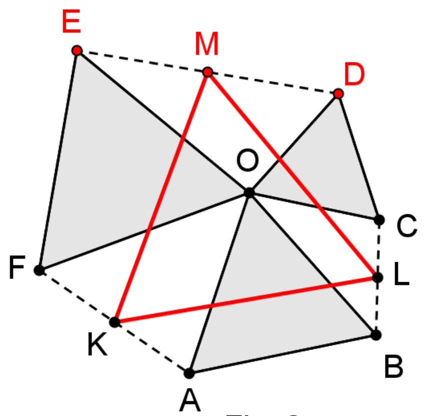
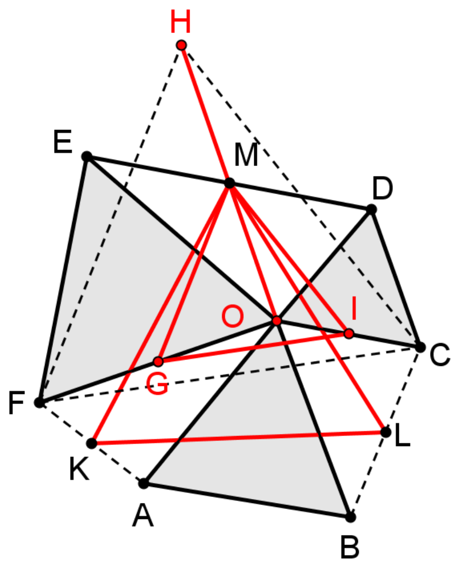


Fig. 2

The following interesting results also arise from the proof:

Creating new relevant propositions
From the 1st underlining, i.e from the relation
$\underline{F \cdot C + A \cdot B} = M$ the following emerge:
$F \cdot C$ is the vertex H of the equilateral triangle FCH and $A \cdot B$ is the point O. So the relation becomes: $H + O = M$ which means that M is the midpoint of OH.



The 2nd underlining, i.e. the relation
$\underline{(F + O) \cdot (O + C)} = M$ results (without the use of the previous observation) in the following:
$F + O$ is the midpoint G of the segment OF and $O + C$ is the midpoint I of the segment OC.
So the relation becomes $G \cdot I = M$ which means that the triangle GIM is equilateral.

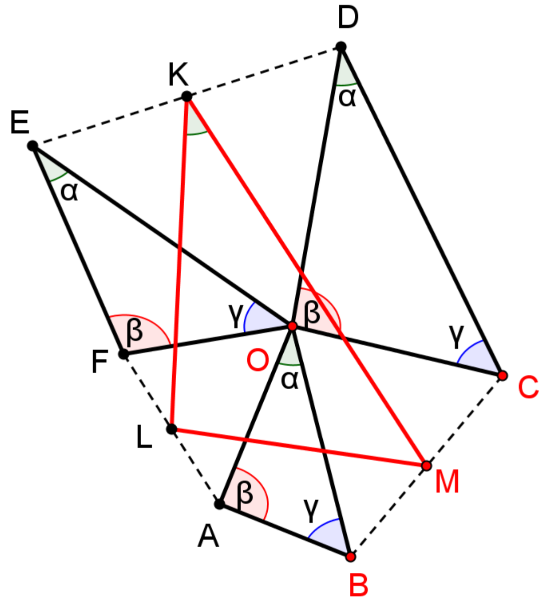
From the relation $K \cdot L = M$ that we proved, if we write it as $(F + A) \cdot (B + C) = M$ or $F \cdot B + A \cdot C = M$ it follows that M is the midpoint of segment that joins the other vertices of the equilateral triangles with sides FB and AC.
And more generally, if in the relation $K \cdot L = M$ we replace points K and L with various combinations of operations we prove other propositions related to the figure.

22) Farther generalization of the equilateral triangle proposition.
The triangles OAB, OCD and OEF of the figure are similar with homologous vertices as in the figure and with the same orientation. If K, L, M are the midpoints of the segments DE, FA and BC respectively, prove that the triangle KLM is similar to OAB.

Proof

$$\text{Let } \frac{OB}{OA} = \frac{DC}{DO} = \frac{EO}{EF} = \lambda$$

Consider the P.O + = (δ, 0) and · = (λ, α).
Then: O·A = B , D·O = C , E·F = O
We will prove that K·L = M



Indeed: K·L = (D + E)·(A + F) = D·A + E·F = D·A + O = D·A + O·O = (D + O)·(A + O) = (D + O)·(O + A) = D·O + O·A = C + B = M

which means that $\frac{KM}{KL} = \lambda$ and

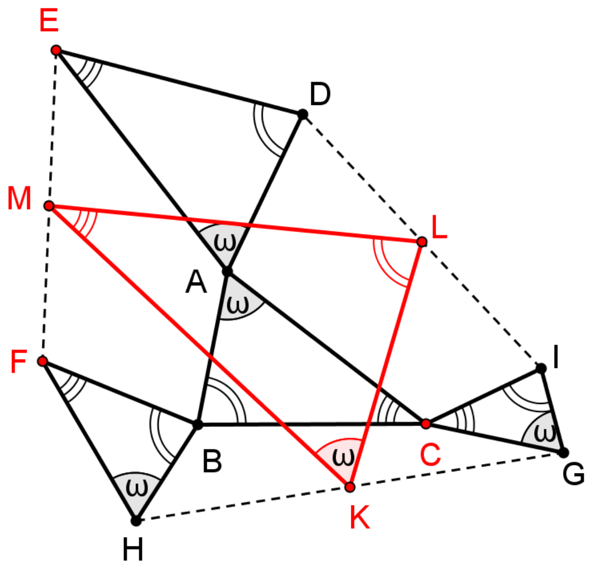
∡LKM= α, so the triangle KLM is similar to OAB.

23) In the following figure the triangles ABC, ADE, HBF, GIC are similar with the same orientation and with equal angles as shown by the figure.
If K, L, M are the midpoints of HG, ID and EF respectively, prove that the triangle KLM is similar to ABC.

Proof

Let ∡BAC = ∡DAE = ∡BHF = ∡IGC = ω and

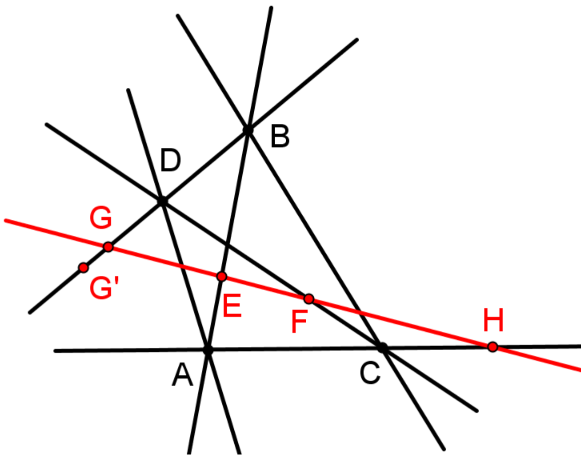
$$\frac{AC}{AB} = \frac{AE}{AD} = \frac{HF}{HB} = \frac{GC}{GI} = \lambda$$



Consider the P.O + = (δ, 0) and · = (λ, ω)
Then
H + G = K, I + D = L, E + F = M
and A·B = C , A·D = E , H·B = F ,
G·I = C
We will prove that K·L = M which means $\frac{KM}{KL} = \lambda$ and ∡LKM = ω
so the triangle KLM is similar to ABC.
Indeed: K·L = (H + G)·(D + I) =
H·D + G·I = H·D + C = H·D + A·B =
(H + A)·(D + B) = (H + A)·(B + D) = H·B + A·D = F + E = M

24) A USEFUL PROPOSITION

If A, B, C, D any points and a line (ε) intersects the lines AB, CD, AC και BD at the points E, F, H, and G respectively so that $\dfrac{\overrightarrow{EA}}{\overrightarrow{EB}}=\dfrac{\overrightarrow{FC}}{\overrightarrow{FD}}$ **then** $\dfrac{\overrightarrow{HA}}{\overrightarrow{HC}}=\dfrac{\overrightarrow{GB}}{\overrightarrow{GD}}$**.**



Proof

$\dfrac{\overrightarrow{EA}}{\overrightarrow{EB}}=\dfrac{\overrightarrow{FC}}{\overrightarrow{FD}}\Rightarrow\dfrac{\overrightarrow{AE}}{\overrightarrow{EB}}=\dfrac{\overrightarrow{CF}}{\overrightarrow{FD}}\Rightarrow$

$\dfrac{\overrightarrow{AE}}{\overrightarrow{AE}+\overrightarrow{EB}}=\dfrac{\overrightarrow{CF}}{\overrightarrow{CF}+\overrightarrow{FD}}\Rightarrow$

$\dfrac{\overrightarrow{AE}}{\overrightarrow{AB}}=\dfrac{\overrightarrow{CF}}{\overrightarrow{CD}}=\lambda$

We consider point G′ of the line BD such that $\dfrac{\overrightarrow{G'B}}{\overrightarrow{G'D}}=\dfrac{\overrightarrow{HA}}{\overrightarrow{HC}}$

We will prove that $G'=G$

$\dfrac{\overrightarrow{G'B}}{\overrightarrow{G'D}}=\dfrac{\overrightarrow{HA}}{\overrightarrow{HC}}\Rightarrow\dfrac{\overrightarrow{AH}}{\overrightarrow{HC}}=\dfrac{\overrightarrow{BG'}}{\overrightarrow{G'D}}\Rightarrow\dfrac{\overrightarrow{AH}}{\overrightarrow{AH}+\overrightarrow{HC}}=\dfrac{\overrightarrow{BG'}}{\overrightarrow{BG'}+\overrightarrow{G'D}}\Rightarrow\dfrac{\overrightarrow{AH}}{\overrightarrow{AC}}=\dfrac{\overrightarrow{BG'}}{\overrightarrow{BD}}=\mu$

Consider the P.O + = (λ, 0) and · = (μ, 0)

The following applies: A + B = E , C + D = F , A·C = H , B·D = G′

However: (A + B)·(C + D) = A·C + B.D ⇒ E·F = H + G′

Because E·F∈EF i.e. E·F∈ε ⇒ H + G′∈ε and because H∈ε ⇒ G′∈ε

So G′∈ε∩BD = G and the proposition is proved.

Note

The above proposition is useful for solving many problems and is easy to memorize as follows: $\dfrac{\square A}{\square B}=\dfrac{\square C}{\square D}\Rightarrow\dfrac{\square A}{\square C}=\dfrac{\square B}{\square D}$

where the squares are the points of divisions of segments AB, CD, AC and BD respectively.

The above implication is reminiscent of the property of proportions,

$\dfrac{A}{B}=\dfrac{C}{D}\Rightarrow\dfrac{A}{C}=\dfrac{B}{D}$

i.e. the change of the middle terms of a proportion.

25) Use more than two P.O

Lemma

Let ABC be any triangle and G be its barycenter.

Let also the P.O + = (δ, 0) and · = $(\frac{2}{3}, 0)$. Then

G = A· (B + Γ) (Fig. 1)

Indeed, if D is the midpoint of BC, then: D = B + C, so G = A·D = A· (B + C)

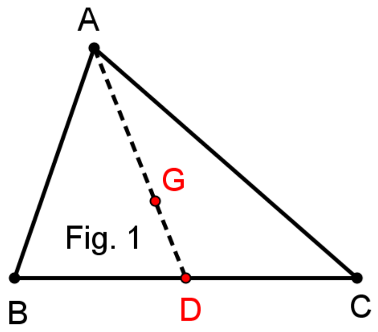


Fig. 1

We will rely on the above lemma to prove the following:

Proposition

Exterior to a triangle ABC we construct the similar triangles CBD, ACE, BAF with the same orientation. Then the triangles ABC and DEF have the same barycenter (Fig. 2).

Proof

Because the triangles CBD, ACE and BAF are similar, there is a P.O such that CoB = D , AoC = E , BoA = F

(This P.O is o = (λ, ω) where $\lambda = \frac{CD}{CB} = \frac{AE}{AC} = \frac{BF}{BA}$ and ω = ∡BCD)

Let M be the midpoint of AB and P be the midpoint of CA.

Let G also be the barycenter of ABC. We will prove that G is the barycenter of the triangle DEF.

Consider the P.O + = (δ, 0) and · = $(\frac{2}{3}, 0)$

Then G = A·(B + C). We must and it is enough to prove that G = D·(E + F)

Indeed,

D·(E + F) = (CoB)·[(AoC) + (BoA)] =

(CoB)·[(A + B)o(C + A)] =

(CoB)·(MoP) = (C·M)o(B·P) = GoG = G

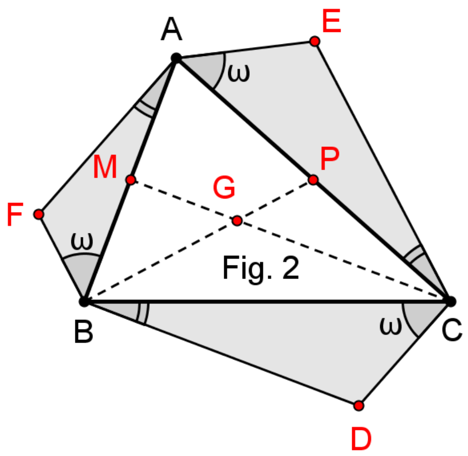


Fig. 2

Corollary

If D_1, E_1, F_1 are homologous points of the triangles CBD, ACE, BAF, then the triangle $D_1E_1F_1$ has the same barycenter as ABC.