RMM - Geometry Marathon 1301 - 1400

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1301. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{w_a^k}{b^2+c^2}\ge\frac{(3r)^k}{2R^2}, k\in\mathbb{N}, k\ge 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{w_a^k}{b^2+c^2}=\sum_{cyc}\frac{\left(w_a^{\frac{k}{2}}\right)^2}{b^2+c^2}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}w_a^{\frac{k}{2}}\right)^2}{2\sum_{cyc}a^2}\overset{\text{Leibnitz}}{\ge}\frac{\left(\sum_{cyc}w_a^{\frac{k}{2}}\right)^2}{18R^2}\overset{?}{\ge}\frac{(3r)^k}{2R^2}$$

$$\Leftrightarrow\sum_{cyc}w_a^{\frac{k}{2}}\underbrace{\overset{?}{\ge}}_{(*)}3(3r)^{\frac{k}{2}}$$

$$\text{Now,}\sum_{cyc}w_a^{\frac{k}{2}}\overset{A-G}{\ge}3.\sqrt[3]{\left(\prod_{cyc}w_a\right)^{\frac{k}{2}}}\overset{?}{\ge}3(3r)^{\frac{k}{2}}\Leftrightarrow\left(\prod_{cyc}w_a\right)^{\frac{k}{2}}\overset{?}{\ge}(27r^3)^{\frac{k}{2}}$$

$$\Leftrightarrow\frac{k}{2}.\ln\left(\frac{\prod_{cyc}w_a}{27r^3}\right)\overset{?}{\ge}0\Leftrightarrow\ln\left(\frac{\prod_{cyc}w_a}{27r^3}\right)\overset{?}{\ge}0\ (\because k\in\mathbb{N},k\ge 2)\Leftrightarrow\prod_{cyc}w_a\underbrace{\overset{?}{\ge}}_{(**)}27r^3$$

$$\text{Again,}\prod_{cyc}w_a\ge\prod_{cyc}h_a=\frac{2r^2s^2}{R}\overset{?}{\ge}27r^3\Leftrightarrow 2s^2\overset{?}{\ge}27Rr\to\text{true}$$

$$\because 2s^2\overset{\text{Gerretsen}}{\ge}27Rr+5r(R-2r)\overset{\text{Euler}}{\ge}27Rr\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\text{ in any }\Delta\text{ ABC},\sum_{cyc}\frac{w_a^k}{b^2+c^2}\ge\frac{(3r)^k}{2R^2}, k\in\mathbb{N}, k\ge 2,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

1302. AD – altitude in acute ΔABC, DE, DF, O_1, O_2 – altitudes and circumcenters in $\Delta ABD, \Delta ACD$. Prove that:

$$4\sqrt{[ABD]\cdot[ACD]}\le\left(\sqrt{DE\cdot DO_2}+\sqrt{DF\cdot DO_1}\right)^2\le\frac{RF}{r}$$

Proposed by Radu Diaconu – Romania

Solution by Tapas Das – India

Since ΔABD is right angle triangle

$\therefore O_1$ = circumcentre of ΔABD = Mid point of AB. Similarly, O_2 = Mid point of AC

$$\Delta ABC = \Delta ABD + ADC.$$

$$F = \frac{1}{2}\cdot C\cdot DE + \frac{1}{2}\cdot b\cdot DF \Rightarrow 2F = C\cdot DE + b\cdot DF$$

$$\therefore DO_1 = \frac{c}{2}, \qquad DO_2 = \frac{b}{2}$$

$$4\sqrt{[ABD]\cdot[ACD]} = 4\sqrt{\tfrac{1}{2}AB\cdot DE\cdot\tfrac{1}{2}AC\cdot DF} = 2\sqrt{C\cdot DE\cdot b\cdot DF} \qquad (1)$$

Now $\left(\sqrt{DE\cdot DO_2} + \sqrt{DF\cdot DO_1}\right)^2 \overset{AM-GM}{\geq} 4\left(\sqrt{DE\cdot DO_2\cdot DF\cdot DO_1}\right)$

$$= 4\sqrt{\tfrac{c}{2}\cdot DE\cdot\tfrac{b}{2}\cdot DF} = 2\sqrt{c\cdot DE\cdot b\cdot DF} \qquad (2)$$

From (1) and (2):

$$4\sqrt{[ABD]\cdot[ACD]} \leq \left(\sqrt{DE\cdot DO_2} + \sqrt{DF\cdot DO_1}\right)^2$$

$$\left[\sqrt{DE\cdot DO_2} + \sqrt{DF\cdot DO_1}\right]^2 == \left[\sqrt{DE\cdot\frac{b}{2}} + \sqrt{DF\cdot\frac{c}{2}}\right]^2 = \left[\sqrt{\frac{DE\cdot c}{2}\cdot\frac{b}{c}} + \sqrt{\frac{DF\cdot b}{2}\cdot\frac{c}{b}}\right]^2$$

$$= \left[\sqrt{[ABD]\cdot\frac{b}{c}} + \sqrt{[ABC]\cdot\frac{c}{b}}\right]^2 \overset{Cauchy-Scwarz}{\leq} ([ABD]+[ADC])\left(\frac{b}{c}+\frac{c}{b}\right) =$$

$$= [ABC]\cdot\left(\tfrac{b}{c}+\tfrac{c}{b}\right) \leq F\cdot\tfrac{R}{r}$$

1303. In ΔABC the following relationship holds:

$$\sum\left(\frac{m_a^2}{bc\cdot\csc\frac{B}{2}}\right)^2 \geq \left(\frac{3r}{2R}\right)^3$$

Proposed by Marin Chirciu – Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{m_a^2}{bc\cdot\csc\frac{B}{2}} \geq \sum\frac{s(s-a)}{bc}\cdot\sin\frac{B}{2} =$$

$$= \sum\cos^2\frac{A}{2}\cdot\sin\frac{B}{2} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod\left(\cos^2\frac{A}{2}\cdot\sin\frac{B}{2}\right)} =$$

$$= 3\cdot\sqrt[3]{\underbrace{\prod\cos^2\frac{A}{2}}_{\left(\frac{s}{4R}\right)^2}\cdot\underbrace{\prod\sin\frac{A}{2}}_{\frac{r}{4R}}} = 3\sqrt[3]{\left(\frac{s}{4R}\right)^2\cdot\frac{r}{4R}} =$$

$$= 3\sqrt[3]{\frac{s^2\cdot r}{(4R)^3}} \overset{s^2\geq 27r^2}{\geq} 3\sqrt[3]{\frac{27r^3}{(4R)^3}} = 3\left(\frac{3r}{4R}\right) \quad (*)$$

$$\sum\left(\frac{m_a^2}{bc\cdot\csc\frac{B}{2}}\right)^2 \overset{(*)}{\geq} 3\left(\frac{3r}{4R}\right)^2 = \frac{3}{4}\left(\frac{3r}{2R}\right)^2 = \frac{3}{4}\left(\frac{3r}{2R}\right)^2 \overset{Euler}{\geq} \frac{3}{4}\cdot\frac{2r}{R}\cdot\left(\frac{3r}{2R}\right)^2 = \left(\frac{3r}{2R}\right)^3$$

Solution 2 by Tapas Das-India

$$\textit{NOTE:- } m_a\cdot m_b\cdot m_c \geq \sqrt{r_ar_b}\cdot\sqrt{r_br_c}\cdot\sqrt{r_cr_a} = (r_ar_br_c) = s^2r$$

$$\textit{NOTE:- } \prod\sin\frac{A}{2} = \frac{r}{4R}$$

$$\text{Now: } \sum\left(\frac{m_a^2}{bc\cdot\csc\frac{B}{2}}\right)^2 \overset{AM-GM}{\geq} 3\left[\left(\frac{m_am_bm_c}{abc}\right)^2\cdot\prod\sin\frac{A}{2}\right]^{\frac{2}{3}} = 3\left[\left(\frac{s^2r}{4RF}\right)^2\cdot\frac{r}{4R}\right]^{\frac{2}{3}}$$

$$\geq 3\left[\left(\frac{s^2\cdot r}{4R\cdot r\cdot s}\right)^2\cdot\frac{r}{4R}\right]^{\frac{2}{3}} = 3\left[\frac{s^2}{16R^2}\cdot\frac{r}{4R}\right]^{\frac{2}{3}} \geq 3\left[\frac{27r^2\cdot r^2}{64R^3}\right] (\because s^2\geq 27r^2)$$

$$= 3\cdot\frac{9r^2}{16R^2} = \frac{27r^2}{16R^2} = \frac{27r^2\cdot r}{16R^2\cdot r} \overset{Euler}{\geq} \frac{27r^3}{16R^2\cdot\frac{R}{2}} = \left(\frac{3r}{2R}\right)^3$$

1304. In any triangle ABC we have the inequality:

$$\left(\tan\frac{\hat{A}}{2}\tan\frac{\hat{B}}{2}\right)^{\tan\frac{\hat{A}}{2}\tan\frac{\hat{B}}{2}}\cdot\left(\tan\frac{\hat{B}}{2}\tan\frac{\hat{C}}{2}\right)^{\tan\frac{\hat{B}}{2}\tan\frac{\hat{C}}{2}}\cdot\left(\tan\frac{\hat{C}}{2}\tan\frac{\hat{A}}{2}\right)^{\tan\frac{\hat{C}}{2}\tan\frac{\hat{A}}{2}} \leq \frac{s^2-2r^2-8Rr}{s^2}$$

Proposed by Radu Diaconu – Romania

Solution by Tapas Das – India

$$\text{Note: } \sum\tan\frac{A}{2} = \frac{4R+r}{s}$$

$$\sum\tan\frac{A}{2}\cdot\tan\frac{B}{2} = 1$$

$$\therefore \left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^{\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\cdot\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^{\tan\frac{B}{2}\cdot\tan\frac{C}{2}}\cdot\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^{\tan\frac{C}{2}\tan\frac{A}{2}} \leq$$

$$\overset{AM-GM}{\leq}\left[\frac{\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\sum\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\right]^{\sum\tan\frac{A}{2}\cdot\tan\frac{B}{2}}=\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}=1-\frac{2r^2+8Rr}{s^2}$$

$$=\frac{s^2-2r^2-8Rr}{s^2}$$

Note:- $\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}=1-\frac{2r^2+8Rr}{s^2}$

1305. In ΔABC the following relationship holds:

$$9\leq\sum\frac{w_a}{h_a}\sum\frac{h_a}{w_a}\leq 9\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

1st part:

$$\sum\frac{w_a}{h_a}\cdot\sum\frac{h_a}{w_a}\overset{Cauchy-Schwarz}{\geq}(1+1+1)^2=9$$

2nd part: Now

$$\sum\frac{w_a}{h_a}\overset{CBS}{\leq}\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{h_a^2}\right)}\leq\sqrt{\left(\sum r_br_c\right)\cdot\frac{\sum a^2}{4F^2}}\overset{Leibniz}{\leq}\sqrt{\frac{s^2 9R^2}{4r^2s^2}}=\frac{3R}{2r}$$

$$\sum\frac{h_a}{w_a}\leq\sum\frac{h_a}{h_a}\ (\because h_a\leq w_a)\ \text{(analog)}=3$$

$$\therefore\sum\frac{w_a}{h_a}\cdot\sum\frac{h_a}{w_a}\leq\frac{3R}{2r}\cdot 3=9\left(\frac{R}{2r}\right)\leq 9\cdot\frac{R\cdot R}{2rR}\overset{Euler}{\leq}9\frac{R^2}{2r\cdot 2r}=9\left(\frac{R}{2r}\right)^2$$

1306. In ΔABC the following relationship holds:

$$\cos A+\cos B-\cos C+1\geq\frac{3\sqrt{3}r^2}{(s-c)R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$\cos A+\cos B-\cos C+1=4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 4\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{4\cdot s\cdot(s-a)(s-b)}{abc} = \frac{4s(s-a)(s-b)(s-c)}{abc(s-c)} = \frac{4\cdot s\cdot sr^2}{abc(s-c)}$$

$$= \frac{4\cdot s^2r^2}{4R\cdot r\cdot s\cdot(s-c)} = \frac{sr}{R(s-c)} \overset{s^2\ge 27r^2}{\ge} \frac{3\sqrt{3}r\cdot r}{R(s-c)} = \frac{3\sqrt{3}r^2}{R(s-c)}$$

Note:

$$\cos A + \cos B - \cos C = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \cos C$$

$$= 2\cos\frac{\pi - C}{2}\cos\frac{A-B}{2} - \cos\left(2\cdot\frac{C}{2}\right) = 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 2\sin^2\frac{C}{2} - 1$$

$$= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\frac{\pi-(A+B)}{2}\right) - 1 = 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right) - 1$$

$$= 2\sin\frac{C}{2}\cdot 2\cos\frac{A}{2}\cdot\cos\frac{B}{2} - 1 = 4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2} - 1$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sin^2\frac{C}{2} = \frac{rc-r}{4R} \quad (1)$$

$$s \ge 3\sqrt{3}r \quad (2)$$

$$\cos A + \cos B - \cos C + 1 \ge \frac{3\sqrt{3}r^2}{(s-c)R}$$

$$\nabla: \sum\cos A + 1 - 2\cos C = \frac{r}{R} + 2\cdot 2\sin^2\frac{C}{2} = \frac{r}{R} + 4\cdot\sin^2\frac{C}{2} \overset{(1)}{=}$$

$$= \frac{r}{R} + 4\cdot\frac{(r_c - r)}{4R} = \frac{r_c}{R} = \frac{F}{(s-c)R} = \frac{s\cdot r}{(s-c)R} \overset{(2)}{\ge} \frac{3\sqrt{3}r^2}{(s-c)R}$$

Solution 3 by Ertan Yildirim-Izmir-Turkiye

Lemma 1: $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Lemma 2: $\tan\frac{A}{2} = \frac{r}{s-a}$

$$\cos A + \cos B - \cos C + 1 = 2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right) + 1 - \cos C$$

$$= 2\cdot\sin\frac{C}{2}\cdot\cos\left(\frac{A-B}{2}\right)+2\sin^2\frac{C}{2}=2\cdot\sin\frac{C}{2}\cdot\left(\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right)$$

$$= 2\cdot\sin\frac{C}{2}\cdot\left(\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right)=2\cdot\sin\frac{C}{2}\cdot 2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}$$

$$= 4\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\cdot\tan\frac{C}{2}=4\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ac}}\cdot\sqrt{\frac{s(s-c)}{ab}}\cdot\frac{r}{s-c}$$

$$= 4\cdot\frac{s\cdot sr}{abc}\cdot r=4\cdot\frac{s\cdot sr}{4R\cdot sr}\cdot\frac{r}{s-c}=\frac{s\cdot r}{(s-c)R}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}r^2}{(s-c)R}$$

Solution 4 by Aissa Hiyab-Morocco

$$\sum\cos A=1+\frac{r}{R}\ \text{(Lemma 1)},\ \sin^2\frac{C}{2}=\frac{c}{s-c}+\frac{r}{4R}\ \text{(Lemma 2)}$$

$$s\geq 3\sqrt{3}r\qquad\text{(Lemma 3)}$$

$$\cos A+\cos B-\cos C+1=\left(\sum\cos A\right)-2\cos C+1$$

$$=\left(1+\frac{r}{R}\right)-2\cos C+1\qquad\text{(Lemma 1)}$$

$$=\frac{r}{R}+2(1-\cos C)=\frac{r}{R}+4\sin^2\frac{C}{2}=\frac{r}{R}+4\times\frac{c}{s-c}\times\frac{r}{4R}\quad\text{(Lemma 2)}$$

$$=\frac{r}{R}\left(1+\frac{c}{s-c}\right)$$

$$=\frac{r}{R}\times\frac{s}{s-c}\overset{Lemma\ 3}{\leq}\frac{r\times 3\sqrt{3}r}{R\times(s-c)}=\frac{3\sqrt{3}r^2}{(s-c)R}$$

Solution 5 by Soumitra Mandal-India

$$\cos A+\cos B-\cos C+1=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin^2\left(\frac{C}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin^2\left(\frac{\pi-A-B}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\cos^2\left(\frac{A+B}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]$$

$$=2\cos\left(\frac{A+B}{2}\right)\cdot 2\cos\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right)\cos\left(\frac{\frac{A+B}{2}-\frac{A-B}{2}}{2}\right)$$

$$= 4\cos\left(\frac{\pi - C}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2} = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 4 \cdot \sqrt{\frac{s(s-a)}{bc}} \cdot \sqrt{\frac{s(s-b)}{ca}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}} = \frac{4}{c} \cdot \frac{s(s-a)(s-b)}{ab}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{abc} \cdot \frac{1}{s-c} = \frac{4\Delta^2}{abc} \cdot \frac{1}{s-c} = \frac{4s^2r^2}{4Rrs} \cdot \frac{1}{s-c} [\because \Delta = sr]$$

$$= \frac{sr}{R(s-c)} \geq \frac{3\sqrt{3}r^2}{R(s-c)} [\because s \geq 3\sqrt{3}r] \text{ (proved)}$$

1307. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \frac{\left(r_b^4 + 2r_br_c\left(r_b^2 + r_c^2\right) + r_c^4\right)^3}{r_b^6 + 2r_br_c\left(r_b^4 + r_c^4\right) + r_c^6} \geq \frac{9.6^7 r^{11}}{81R^5 - 2560r^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Tapas Das-India

$$\sum\left(r_a^4 + 2r_br_a\left(r_a^2 + r_b^2\right) + r_b^4\right) = \left[2\left(\sum r_a^3\right)\left(\sum r_a\right)\right]$$

$$\sum\left[r_a^6 + 2r_br_a\left(r_a^4 + r_b^4\right) + r_b^6\right] = \left[2\left(\sum r_a^4\right)\left(\sum r_a\right)\right]$$

$$\sum \frac{\left(r_a^4 + 2r_ar_b\left(r_a^2 + r_b^2\right) + r_b^4\right)^3}{r_a^6 + 3r_ar_b(r_a^4 + r_b^4) + r_b^6} \overset{Holder}{\geq} \frac{1}{3} \cdot \frac{[2(\sum r_a^3) \cdot (\sum r_a)]^3}{2(\sum r_a^5)(\sum r_a)} = \frac{4}{3} \cdot \frac{(\sum r_a^3)^3 \cdot (\sum r_a)^2}{\sum r_a^5}$$

$$\overset{AM-GM}{\geq} \frac{4}{3} \cdot \frac{(3r_ar_br_c)^3(4R + r)^2}{\sum r_a^5} \overset{Euler}{\geq} \frac{4}{3} \cdot \frac{27(s^2r)^3(9r)^2}{\sum r_a^5}$$

$$\overset{Mitrinovic}{\geq} \frac{4}{3} \cdot \frac{27(27r^3)^3(9)^2 \cdot r^2}{\sum r_a^5} \geq \frac{4}{3} \cdot \frac{27 \cdot 3^9 \cdot r^9 \cdot 9^2 \cdot r^2}{\frac{3^6}{32}(81R^5 - 2560r^5)}$$

$$= (9) \cdot \frac{279936 \cdot r^{11}}{(81R^5 - 2560r^5)} = \frac{9 \cdot 6^7 \cdot r^{11}}{81R^5 - 2560r^5}$$

Now let $a = x + y + z, b = xy + yz + zx, c = xyz$

$$\therefore \sum x^5 = a^5 - 5a^3b + 5ab^2 + 5a^2c - 5bc$$

$$\Rightarrow \sum x^5 = \left(\sum x\right)^5 - 5(x + y)(y + z)(z + x)(x^2 + y^2 + z^2 + xy + yz + zx)$$

$$\therefore \Rightarrow \sum r_a^5 = \left(\sum r_a\right)^5 - 5(r_a + r_b)(r_b + r_c)(r_c + r_a)$$

$$\overset{AM-GM}{\geq} = (4R+r)^5 - 5\cdot(8\cdot r_a r_b r_c)\left(2\cdot 3(r_a r_b r_c)^{\frac{2}{3}}\right)$$

$$\overset{Mitrinovic}{=} (4R+r)^5 - 5\cdot(8\cdot s^2 r)\left(2\cdot 3(s^2 r)^{\frac{2}{3}}\right) \overset{Euler}{\leq} \left(\frac{9R}{2}\right)^5 - 5(8\cdot 3^3 r^3)(2\cdot 3^3 r^2)$$

$$= 3^6\frac{(81R^5 - 2560r^5)}{32}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\left(r_b^4 + 2r_b r_c(r_b^2 + r_c^2) + r_c^4\right)^3}{r_b^6 + 2r_b r_c(r_b^4 + r_c^4) + r_c^6} \overset{Holder}{\geq} \frac{8\left(\sum_{cyc} r_a^4 + \sum_{cyc} r_b r_c(r_b^2 + r_c^2)\right)^3}{6\left(\sum_{cyc} r_a^6 + \sum_{cyc} r_b r_c(r_b^4 + r_c^4)\right)}$$

$$= \frac{4\left(\sum_{cyc} r_a^3\right)^3\left(\sum_{cyc} r_a\right)^3}{3\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_a^5\right)} \overset{Reverse\ Chebyshev}{\geq} \frac{4\left(\sum_{cyc} r_a^3\right)^3\left(\sum_{cyc} r_a\right)^3}{9\left(\sum_{cyc} r_a^6\right)} \overset{Holder}{\geq} \frac{4\left(\sum_{cyc} r_a\right)^9\left(\sum_{cyc} r_a\right)^3}{9^4\left(\sum_{cyc} r_a^6\right)}$$

$$= \frac{4(4R+r)^9(4R+r)^3}{9^4\left(\sum_{cyc} r_a^6\right)} \overset{Euler}{\geq} \frac{4.9^9.r^9.(4R+r)^3}{9^4\left(\sum_{cyc} r_a^6\right)} \overset{?}{\geq} \frac{9.6^7 r^{11}}{81R^5 - 2560r^5}$$

$$= \frac{3^9.2^7.r^{11}}{81R^5 - 2560r^5} \Leftrightarrow \frac{3(81R^5 - 2560r^5)(4R+r)^3}{32r^2} \underset{(*)}{\overset{?}{\geq}} \sum_{cyc} r_a^6$$

$$\text{Now, } \sum_{cyc} r_a^6 = \left(\sum_{cyc} r_a^2\right)^3 - 3\prod_{cyc}(r_b^2 + r_c^2) \overset{Cesaro}{\leq} \left((4R+r)^2 - 2s^2\right)^3 - 24r_a^2 r_b^2 r_c^2$$

$$\overset{Gerretsen}{\leq} \left((4R+r)^2 - 2(16Rr - 5r^2)\right)^3 - 6r^2(2s^2)^2 \overset{Gerretsen}{\leq}$$

$$(16R^2 - 24Rr + 11r^2)^3 - 6r^2\left(27Rr + 5r(R-2r)\right)^2$$

$$\left(\begin{array}{c}\because 16R^2 - 24Rr + 11r^2 = 16R(R-2r) \\ +8Rr + 11r^2 \overset{Euler}{\geq} 8Rr + 11r^2 > 0\end{array}\right) \overset{Euler}{\leq} (16R^2 - 24Rr + 11r^2)^3$$

$$-6r^2(27Rr)^2 \overset{?}{\leq} \frac{3(81R^5 - 2560r^5)(4R+r)^3}{32r^2}$$

$$\Leftrightarrow 15552t^8 + 11664t^7 - 128156t^6 + 590067t^5 - 1155072t^4 + 761856t^3$$

$$-1022784t^2 + 186624t - 50272 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(\begin{array}{c}15552t^6 + 73872t^5 + 105124t^4 + 715075t^3 \\ +1284732t^2 + 3040484t + 6000224\end{array}\right) + 12025584\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{Euler}{\geq} 2 \Rightarrow (*) \text{ is true } \therefore \text{ in any } \Delta ABC, \sum_{cyc}\frac{\left(r_b^4 + 2r_b r_c(r_b^2 + r_c^2) + r_c^4\right)^3}{r_b^6 + 2r_b r_c(r_b^4 + r_c^4) + r_c^6}$$

$$\geq \frac{9.6^7 r^{11}}{81R^5 - 2560r^5}, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1308. If $x,y,z>0$ and $n\in\mathbb{N}$, then in ΔABC holds:

$$\frac{a^{4n-1}\cdot x^{2n}}{h_a}+\frac{b^{4n-1}\cdot y^{2n}}{h_b}+\frac{c^{4n-1}\cdot z^{2n}}{h_c}\geq\frac{3}{2F}\left(\frac{16}{9}F^2\sum yz\right)^n$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\frac{a^{4n-1}\cdot x^{2n}}{h_a}+\frac{b^{4n-1}\cdot y^{2n}}{h_b}+\frac{c^{4n-1}\cdot z^{2n}}{h_c}=\frac{a^{4n}x^{2n}}{2F}+\frac{b^{4n}y^{2n}}{2F}+\frac{c^{4n}z^{2n}}{2F}$$

$$(\because h_a=\frac{2F}{a})$$

$$=\frac{1}{2F}[(a^2x)^{2n}+(b^2y)^{2n}+(c^2z)^{2n}]\overset{CBS}{\geq}\frac{1}{2F}\cdot\frac{1}{3^{2n-1}}(a^2x+b^2y+c^2z)^{2n}$$

$$\geq\frac{1}{2F}\cdot\frac{1}{3^{2n-1}}\left(4F\sqrt{xy+yz+zx}\right)^{2n}\text{(Oppenheim)}$$

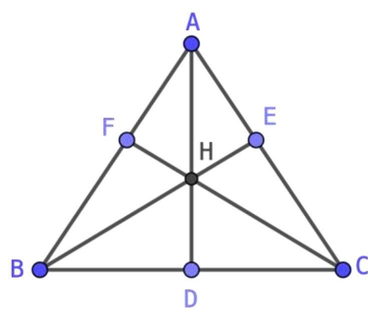
$$=\frac{1}{2F}\cdot\frac{3}{3^{2n}}\left(16F^2\left(\sum xy\right)\right)^n=\frac{3}{2F}\left(\frac{16}{9}\left(\sum xy\right)\right)^n$$

1309. $\Delta DEF,\Delta XYZ$ – are the orthic and the circumcevian triangle of altitudes in acute ΔABC. Prove that:

$$6RF\cdot\sqrt[3]{\frac{2r}{R}\cdot\prod_{cyc}\cos(B-C)}\leq\sum_{cyc}AD\cdot AX\cdot(s-a)\leq s(s^2+r^2-8Rr)$$

Proposed by Radu Diaconu – Romania

Solution by Tapas Das – India



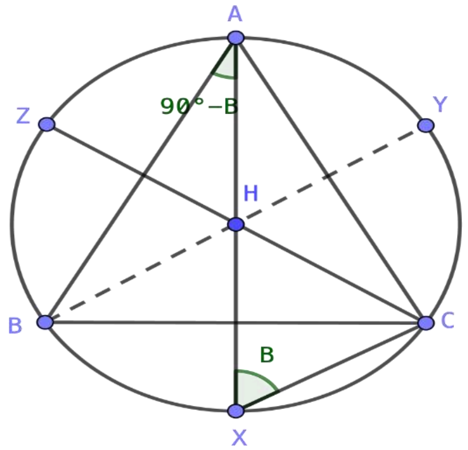
From ΔABD we get

$$\sin B=\frac{AD}{AB}$$

$$\sin B = \frac{AD}{C}$$

$$\therefore AD = c\sin B$$

(analog)



$$\angle BAX = \angle BCX = 90^\circ - B$$

(angles are on the same arc)

$$\therefore \angle ACX = C + 90^\circ - B = 90^\circ - (B - C)$$

From ΔACX we get:

$$\frac{AX}{\sin\angle ACX} = \frac{AC}{\sin\angle AXC}$$

$$[\angle AXC = \angle ABC = B] \Rightarrow \frac{AX}{\sin[90^\circ - (B - C)]} = \frac{b}{\sin B}$$

$$\therefore AX = \frac{b\cos(B-C)}{\sin B} \quad \textbf{(analog)}$$

$$\therefore AD\cdot AX\cdot(s-a) = c\sin B\cdot\frac{b\cos(B-C)}{\sin B}\cdot(s-a) = bc(s-a)\cos(B-C)$$

RHS

$$\therefore \sum AD\cdot AX\cdot(s-a) = \sum bc(s-a)\cos(B-C) \le bc(s-a)\cdot 1$$

[Note: $\cos(B-C) \le 1$]

$$= s\left(\sum ab\right) - 3abc = s(s^2 + r^2 + 4Rr) - 12Rrs = s[s^2 + r^2 - 8Rr]$$

LHS

$$\sum AD\cdot AX\cdot(s-a) = \sum bc\,(s-a)\cos(B-C)$$

$$= abc\sum\frac{(s-a)}{a}\cdot\cos(B-C)$$

$$\overset{AM-GM}{\ge} 3abc\left[\frac{(s-a)(s-b)(s-c)}{abc}\cdot\prod\cos(B-C)\right]^{\frac{1}{3}}$$

$$\ge 12RF\left[\frac{sr^2}{4Rrs}\prod\cos(B-C)\right]^{\frac{1}{3}} = 12RF\left[\frac{r}{4R}\prod\cos(B-C)\right]^{\frac{1}{3}}$$

$$=12RF\left[\frac{2r}{8R}\prod\cos(B-C)\right]^{\frac{1}{3}}=\frac{12RF}{2}\left[\frac{2r}{R}\prod\cos(B-C)\right]^{\frac{1}{3}}$$

$$=6RF\sqrt[3]{\frac{2r}{R}\prod\cos(B-C)}$$

1310. In ΔABC the following relationship holds:

$$\sum\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{2}+\frac{1}{3\sqrt{3}}\cdot\frac{R}{2r}\sum\sin\frac{A}{2}\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\text{Let } f(x)=\sin\frac{x}{2}, x\in(0,\pi)$$

$$\therefore f'(x)=\frac{1}{2}\cos\frac{x}{2},\qquad f''(x)=-\frac{1}{4}\sin\frac{x}{2}<0$$

$\therefore f$ is concave, using Jensen's

$$\therefore\frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}{3}\le\sin\left(\frac{A+B+C}{6}\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

$$\therefore\sum\sin\frac{A}{2}\le\frac{3}{2}$$

$$\therefore\sum\sin\frac{A}{2}\sin\frac{B}{2}\le\frac{\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2}{3}\le\frac{\left(\frac{3}{2}\right)^2}{3}=\frac{3}{4}\qquad\textbf{(1)}$$

$$\left[\text{Note: }\sum xy\le\frac{(\sum x)^2}{3}\right]$$

Now,

$$\sum\sin\frac{A}{2}\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\ge}3\left[\prod\sin\frac{A}{2}\cdot\prod\cos\frac{A}{2}\right]^{\frac{1}{3}}=3\left[\frac{r}{4R}\cdot\frac{s}{4R}\right]^{\frac{1}{3}}$$

$$\ge3\left[\frac{r}{4R}\cdot\frac{3\sqrt{3}r}{4R}\cdot\frac{4R}{4R}\right]^{\frac{1}{3}}(\because s\ge3\sqrt{3}r)\overset{Euler}{\ge}3\left[\frac{r}{4R}\cdot\frac{3\sqrt{3}r}{4R}\cdot\frac{8r}{4R}\right]^{\frac{1}{3}}=\frac{3\sqrt{3}r\cdot2}{4R}$$

$$\therefore\frac{1}{2}+\frac{1}{3\sqrt{3}}\cdot\frac{R}{2r}\sum\sin\frac{A}{2}\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}\ge\frac{1}{2}+\frac{1}{3\sqrt{3}}\cdot\frac{R}{2r}\cdot\frac{3\sqrt{3}r\cdot2}{4R}=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}\qquad\textbf{(2)}$$

From (1) and (2) we get the desired result.

1311. Prove that in any triangle ABC holds the inequality

$$\left(\frac{a}{2m_a}\right)^2+\left(\frac{b}{2m_b}\right)^2+\left(\frac{c}{2m_c}\right)^2\geq 1$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Tapas Das-India

$$\text{Let } a>b>c$$

$$\therefore m_a<m_b<m_c$$

$$\therefore \left(\frac{a}{2m_a}\right)^2+\left(\frac{b}{2m_b}\right)^2+\left(\frac{c}{2m_c}\right)^2\geq$$

$$\overset{Chebyshev}{\geq}\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)$$

$$\geq\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\cdot\frac{(1+1+1)^2}{m_a^2+m_b^2+m_c^2}=\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\cdot\frac{9}{\frac{3}{4}(a^2+b^2+c^2)}$$

$$=\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\cdot\frac{3\times 4}{(a^2+b^2+c^2)}=1$$

1312. In any acute Δ ABC, the following relationship holds :

$$8\prod_{cyc}\frac{m_a s_a}{h_a r_a}\leq\prod_{cyc}\frac{r_a+r_b}{r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now,}\frac{m_a^2}{s(s-a)}-1\overset{?}{\leq}\frac{b^2+c^2}{2bc}-1\Leftrightarrow\frac{(b-c)^2+4s(s-a)-4s(s-a)}{4s(s-a)}\overset{?}{\leq}\frac{(b-c)^2}{2bc}$$

$$\Leftrightarrow(b-c)^2.\left(\frac{1}{2bc}-\frac{1}{4s(s-a)}\right)\overset{?}{\geq}0\Leftrightarrow(b-c)^2.\left(\frac{4s(s-a)-2bc}{8sbc(s-a)}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow(b-c)^2.\frac{b^2+c^2-a^2}{8sbc(s-a)}\overset{?}{\geq}0\Leftrightarrow(b-c)^2.\frac{\cos A}{4s(s-a)}\overset{?}{\geq}0\to\text{true}\because\cos A>0$$

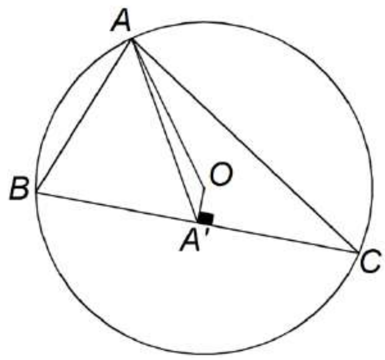
in acute triangles $\Rightarrow \frac{m_a^2}{s(s-a)} \le \frac{b^2+c^2}{2bc} \Rightarrow m_a^2 \cdot \frac{2bc}{b^2+c^2} \le s(s-a)$

$$\Rightarrow \frac{m_a s_a}{h_a r_a} = \frac{m_a^2 \cdot \frac{2bc}{b^2+c^2}}{h_a r_a} \le \frac{s(s-a)}{h_a r_a} = \frac{2R.s(s-a)}{bcr_a} \overset{\text{via (i)}}{=} \frac{r_b+r_c}{2r_a} \Rightarrow \frac{2m_a s_a}{h_a r_a} \le \frac{r_b+r_c}{r_a}$$

and analogs $\Rightarrow 8\prod_{cyc}\frac{m_a s_a}{h_a r_a} \le \prod_{cyc}\frac{r_b+r_c}{r_a} = \prod_{cyc}\frac{r_a+r_b}{r_c}$ $\therefore$ in any acute Δ ABC,

$$8\prod_{cyc}\frac{m_a s_a}{h_a r_a} \le \prod_{cyc}\frac{r_a+r_b}{r_c}, \text{"="} \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

Solution 2 by Mohamed Amine-Tanger-Morocco



Let A′ be the midpoint of BC and O the circumcenter of ΔABC. In $\Delta AA'O$, we have

$$m_a = AA' \le AO + OA' = R + R\cos A = R(1+\cos A) = 2R\cos^2\frac{A}{2}.$$

Using this result, we have

$$\frac{h_a}{m_a s_a} = \frac{bc}{2R}\cdot\frac{b^2+c^2}{2bcm_a^2} = \frac{b^2+c^2}{R.4m_a^2} = \frac{1}{2R}\left(1+\frac{a^2}{4m_a^2}\right) \ge \frac{1}{2R}\left(1+\frac{\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)^2}{4\left(2R\cos^2\frac{A}{2}\right)^2}\right)$$

$$= \frac{1}{2R}\left(1+\tan^2\frac{A}{2}\right) = \frac{1}{2R\cos^2\frac{A}{2}} = \frac{bc}{2Rs(s-a)} = \frac{2r(s-b)(s-c)}{a.sr^2} = \frac{2}{\frac{sr}{s-b}+\frac{sr}{s-c}} = \frac{2}{r_b+r_c}.$$

Thus,

$\frac{2m_a s_a}{h_a r_a} \le \frac{r_b+r_c}{r_a}$. Multiplying this inequality with similar ones yields the desired result.

Equality holds if and only if ΔABC is equilateral.

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$8\cdot\prod\frac{m_a\cdot s_a}{h_a\cdot r_a} \le \prod\frac{r_b+r_c}{r_a},\ b^2+c^2 \le 4R\cdot m_a \quad (1)$$

$$m_a \leq 2R\cos^2\frac{A}{2}$$

$$4R\cdot\cos^2\frac{A}{2} = \frac{2a}{\sin A}\cdot\cos^2\frac{A}{2} = a\cdot\cot\frac{A}{2} = \frac{a\cdot F}{(s-b)(s-c)} =$$

$$= \frac{(s-b+s-c)\cdot F}{(s-b)(s-c)} = \frac{F}{s-b} + \frac{F}{s-c} = r_b + r_c$$

$$4R\cdot\cos^2\tfrac{A}{2} = r_b + r_c \quad (3)$$

$$\Rightarrow \frac{2\cdot m_a\cdot s_a}{h_a\cdot r_a} = \frac{2m_a\cdot\frac{2bc}{b^2+c^2}\cdot m_a}{h_a\cdot r_a} = \frac{4bc}{h_a}\cdot\frac{m_a^2}{(b^2+c^2)r_a} =$$

$$= \frac{8R}{r_a}\left(\frac{2(b^2+c^2)-a^2}{4(b^2+c^2)}\right) = \frac{2R}{r_a}\cdot\left(2-\frac{a^2}{b^2+c^2}\right) \overset{(1)}{\leq}$$

$$\leq \frac{2R}{r_a}\cdot\left(2-\frac{a^2}{4R\cdot m_a}\right) \overset{(2)}{\leq} \frac{2R}{r_a}\cdot\left(2-\frac{a^2}{8R^2\cdot\cos^2\frac{A}{2}}\right) =$$

$$= \frac{2R}{r_a}\left(2-\frac{\sin^2\frac{A}{2}}{2\cos^2\frac{A}{2}}\right) = \frac{2R}{r_a}\left(2-2\sin^2\frac{A}{2}\right) = \frac{4R\cdot\cos^2\frac{A}{2}}{r_a} \overset{(3)}{=} \frac{r_b+r_c}{r_a}$$

1313. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left(\frac{n_a^2}{w_a^2}+\frac{4r_a}{r_b+r_c}\right) \geq 1+\frac{4R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2$$

$$\Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$

$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= as^2 - 2ah_ar_a \Rightarrow n_a^2 = s^2 - 2h_ar_a \text{ and analogs} \Rightarrow \sum_{cyc}\left(\frac{n_a^2}{w_a^2} + \frac{4r_a}{r_b + r_c}\right)$$

$$= s^2\sum_{cyc}\frac{1}{w_a^2} - \sum_{cyc}\frac{2h_ar_a}{w_a^2} + 4\left(\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{r_b + r_c}\right) - 3\right) \overset{\text{via (i)}}{=}$$

$$s^2\sum_{cyc}\frac{(b+c)^2a}{4abcs(s-a)} - \sum_{cyc}\frac{4r^2s^2(b+c)^2}{a(s-a).4bcs(s-a)} + 4\left((4R+r)\sum_{cyc}\frac{1}{4R\cos^2\frac{A}{2}} - 3\right)$$

$$= \frac{s^2}{16Rrs^2}\sum_{cyc}\left(\frac{a\left(s^2 + (s-a)^2 + 2s(s-a)\right)}{s-a}\right)$$

$$-\frac{r^2s^2}{4Rrs^2}.\sum_{cyc}\frac{s^2 + (s-a)^2 + 2s(s-a)}{(s-a)^2} + \frac{(4R+r)\left(s^2 + (4R+r)^2\right) - 12Rs^2}{Rs^2}$$

$$= \frac{1}{16Rr}\left(s^2\sum_{cyc}\frac{a - s + s}{s-a} + s(2s) - 2\left(s^2 - 4Rr - r^2\right) + 2s.2s\right)$$

$$-\frac{r}{4R}\left(\frac{1}{r^2}\sum_{cyc} r_a^2 + 3 + \frac{2s\left(4Rr + r^2\right)}{r^2s}\right) + \frac{(4R+r)\left(s^2 + (4R+r)^2\right) - 12Rs^2}{Rs^2}$$

$$= \frac{1}{16Rr}\left(s^2\left(-3 + \frac{s\left(4Rr + r^2\right)}{r^2s}\right) + 2\left(4Rr + r^2\right) + 4s^2\right)$$

$$-\frac{r}{4R}\left(\frac{(4R+r)^2 - 2s^2 + 3r^2}{r^2} + \frac{2(4R+r)}{r}\right) + \frac{(4R+r)\left(s^2 + (4R+r)^2\right) - 12Rs^2}{Rs^2}$$

$$= \frac{(2R+5r)s^4 - rs^2\left(32R^2 + 92Rr + 3r^2\right) + 8r^2(4R+r)^3}{8Rr^2s^2} \geq 1 + \frac{4R}{r} = \frac{4R+r}{r}$$

$$\Leftrightarrow (2R+5r)s^4 - rs^2\left(64R^2 + 100Rr + 3r^2\right) + 8r^2(4R+r)^3 \overset{(*)}{\geq} 0 \text{ and}$$

$$\because (2R+5r)\left(s^2 - 16Rr + 5r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \geq (2R+5r)\left(s^2 - 16Rr + 5r^2\right)^2$$

$$\Leftrightarrow (40R - 53r)s^2 \overset{(**)}{\geq} r\left(576R^2 - 846Rr + 117r^2\right) \text{ and again, LHS of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$(40R - 53r)\left(16Rr - 5r^2\right) \overset{?}{\geq} r\left(576R^2 - 846Rr + 117r^2\right)$$

$$\Leftrightarrow 32R^2 - 101Rr + 74r^2 \overset{?}{\geq} 0 \Leftrightarrow (32R - 37r)(R - 2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \text{in any } \Delta\, ABC, \sum_{cyc}\left(\frac{n_a^2}{w_a^2} + \frac{4r_a}{r_b + r_c}\right) \geq 1 + \frac{4R}{r},$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have $w_a{}^2 = \frac{4bc}{(b+c)^2}.r_br_c \leq r_br_c.$

Also, we have $n_a{}^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s[a^2-(b-c)^2]}{a}$

$$= s^2 - \frac{4s(s-b)(s-c)}{a}$$

$= s^2 - \frac{4s.sr^2}{a(s-a)} = s^2 - 2h_a r_a$, then $n_a{}^2 + 2h_a r_a = s^2$.

And since $\frac{1}{r_b} + \frac{1}{r_c} = \frac{a}{F} = \frac{2}{h_a}$, then we have $r_b + r_c = \frac{2r_b r_c}{h_a}$.

Using these results, we have :

$$\frac{n_a{}^2}{w_a{}^2} + \frac{4r_a}{r_b + r_c} \geq \frac{n_a{}^2}{r_b r_c} + \frac{2h_a r_a}{r_b r_c} = \frac{s^2}{r_b r_c} = \frac{r_a}{r} \quad \text{(and analogs)}$$

Therefore,

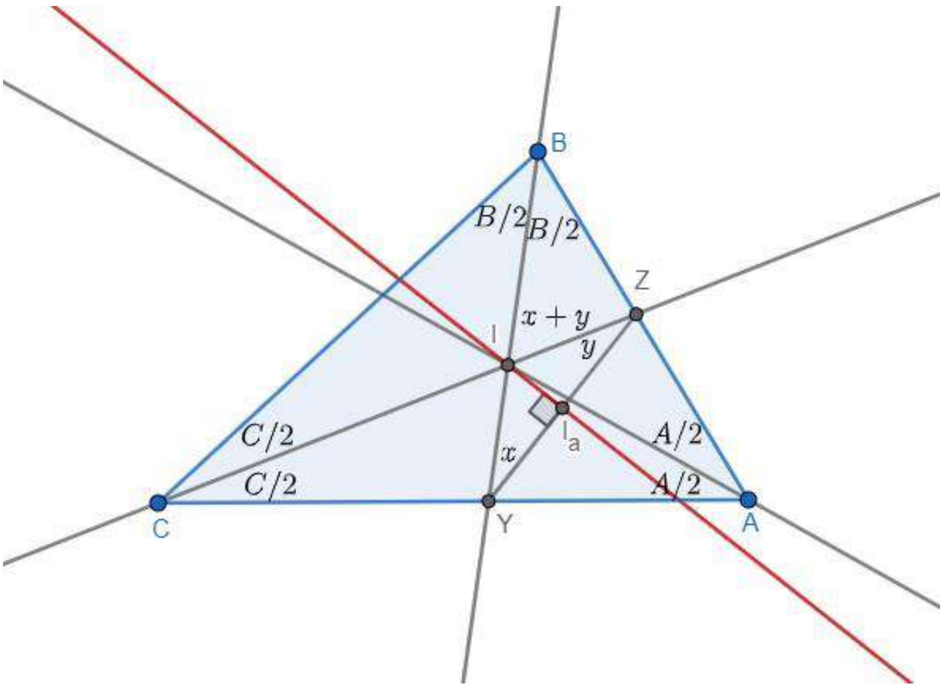
$$\sum_{cyc}\left(\frac{n_a{}^2}{w_a{}^2} + \frac{4r_a}{r_b + r_c}\right) \geq \sum_{cyc}\frac{r_a}{r} = 1 + \frac{4R}{r}.$$

Equality holds if and only if ΔABC is equilateral.

1314.

In Δ ABC picturised in the diagram, the following relationship holds

$$2\sqrt{2}\cdot \mathrm{II}_a < IA$$



Proposed by Aissa Hiyab-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via sine law on } \Delta\, AIY, \frac{AI}{\sin\left(C+\frac{B}{2}\right)} = \frac{YI}{\sin\frac{A}{2}}$$

$$\Rightarrow \frac{AI}{\sin\frac{180^\circ - A + C}{2}} = \frac{YI}{\sin\frac{A}{2}} \Rightarrow YI \overset{(i)}{=} \frac{AI \sin\frac{A}{2}}{\cos\frac{A-C}{2}}$$

$$\text{Via sine law on } \Delta\, AIZ, \frac{AI}{\sin\left(B+\frac{C}{2}\right)} = \frac{ZI}{\sin\frac{A}{2}} \Rightarrow \frac{AI}{\sin\frac{180^\circ - A + B}{2}} = \frac{ZI}{\sin\frac{A}{2}} \Rightarrow ZI \overset{(ii)}{=} \frac{AI \sin\frac{A}{2}}{\cos\frac{A-B}{2}}$$

$$\text{Via } \Delta\, II_aY, II_a = YI \sin x \text{ and via } \Delta\, II_aZ, II_a = ZI \sin y \therefore II_a^2 = YI.ZI.\sin x \sin y$$

$$\overset{\text{via (i),(i)}}{\Rightarrow} 8II_a^2 = 8 \sin x \sin y . \frac{AI^2 \sin^2\frac{A}{2}}{\cos\frac{A-C}{2}.\cos\frac{A-B}{2}} \overset{?}{<} AI^2$$

$$\Leftrightarrow (\cos(x-y) - \cos(x+y)).8\sin^2\frac{A}{2} \overset{?}{<} \cos\frac{2A-(180^\circ - A)}{2}$$

$$+\cos\frac{B-C}{2}\left(\because \frac{-\pi}{2} < \frac{A-C}{2}, \frac{A-B}{2} < \frac{\pi}{2} \Rightarrow \cos\frac{A-C}{2}, \cos\frac{A-B}{2} > 0\right)$$

$$\Leftrightarrow \left(\cos(x-y) - \cos\frac{B+C}{2}\right).8\sin^2\frac{A}{2} \overset{?}{<} \sin\frac{3A}{2} + \frac{b+c}{a}.\sin\frac{A}{2}$$

$$= 3\sin\frac{A}{2} - 4\sin^3\frac{A}{2} + \frac{b+c}{a}.\sin\frac{A}{2}$$

$$\Leftrightarrow \left(\cos(x-y) - \cos\frac{B+C}{2}\right).8\sin\frac{A}{2} \overset{?}{<} 3 - 4\sin^2\frac{A}{2} + \frac{b+c}{a}$$

$$\Leftrightarrow 8\sin\frac{A}{2}.\cos(x-y) - 8\sin^2\frac{A}{2} \overset{?}{<} 3 - 4\sin^2\frac{A}{2} + \frac{b+c}{a}$$

$$\Leftrightarrow \boxed{8\sin\frac{A}{2}.\cos(x-y) \underset{(*)}{\overset{?}{<}} 3 + 4\sin^2\frac{A}{2} + \frac{b+c}{a}}$$

$$\text{Now}, x < x + y = \frac{B+C}{2} = 90^\circ - \frac{A}{2} < 90^\circ \text{ and similarly}, y < 90^\circ \therefore 0 < x < 90^\circ$$

$$\text{and } -90^\circ < -y < 0 \Rightarrow -90^\circ < x - y < 90^\circ \therefore 0 < \cos(x-y) < 1$$

$$\Rightarrow 8\sin\frac{A}{2}.\cos(x-y) < 8\sin\frac{A}{2} \Rightarrow \text{LHS of } (*) \overset{(\bullet)}{<} 8\sin\frac{A}{2} \text{ and } 3 + 4\sin^2\frac{A}{2} + \frac{b+c}{a}$$

$$> 3 + 4\sin^2\frac{A}{2} + 1 \Rightarrow \text{RHS of } (*) \overset{(\bullet\bullet)}{>} 4 + 4\sin^2\frac{A}{2} \therefore (\bullet), (\bullet\bullet) \Rightarrow \text{in order}$$

$$\text{to prove } (*), \text{it suffices to prove} : 4 + 4\sin^2\frac{A}{2} > 8\sin\frac{A}{2} \Leftrightarrow 4\left(1 - \sin\frac{A}{2}\right)^2 > 0$$

$$\rightarrow \text{true} \Rightarrow (*) \text{ is true} \Rightarrow 8II_a^2 < AI^2 \Rightarrow 2\sqrt{2} * II_a < IA \ (QED)$$

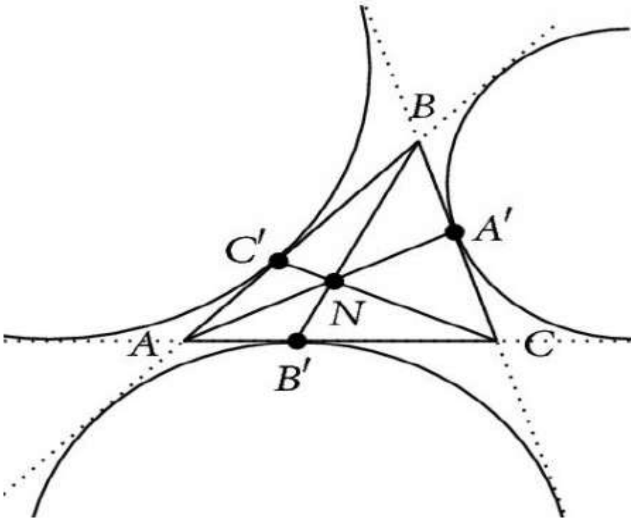
1315. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \frac{n_a}{h_a}\right)^2 . \sum_{cyc} (s - n_b)(s - n_c) > 4s^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let N be the Nagel's point of ΔABC and AA', BB', CC' be the Nagel's cevians.



Since $AB' = s-c,\ \ AC' = s-b,\ \ BC' = CB' = s-a,$

and using Van Aubel's theorem, we have

$$\frac{NA}{NA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$$

$$\Rightarrow \ \frac{n_a}{NA} = 1 + \frac{NA}{NA'} = 1 + \frac{s-a}{a} = \frac{s}{a} \ \Rightarrow \ NA = \frac{an_a}{s} = 2r.\frac{n_a}{h_a} \ \text{(and analogs)}.$$

Also, in ΔNBC, we have

$a < NB + NC = 2r\left(\frac{n_b}{h_b} + \frac{n_c}{h_c}\right)$ **(and analogs), (see, for example, Bogdan Fuștei − *About a Few Special Triangles* − www.ssmrmh.ro)**

Now, we have

$$s^2 - {n_a}^2 = s^2 - \left(s(s-a) + \frac{s(b-c)^2}{a}\right) = \frac{s[a^2-(b-c)^2]}{a} = \frac{4s(s-b)(s-c)}{a}$$

$$= \frac{4s.sr^2}{a(s-a)} = 2r_a h_a.$$

Using these results, we have,

$$s - n_a = \frac{s^2 - {n_a}^2}{s + n_a} = \frac{2r_a h_a}{s + n_a} = \frac{2r_a}{\frac{a}{2r} + \frac{n_a}{h_a}} > \frac{2r_a}{\left(\frac{n_b}{h_b} + \frac{n_c}{h_c}\right) + \frac{n_a}{h_a}}$$

$$\Rightarrow \left(\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\right)(s-n_a) > 2r_a \text{ (and analogs).}$$

Therefore

$$\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2.\sum_{cyc}(s-n_b)(s-n_c) \geq \sum_{cyc}2r_b.2r_c = 4s^2.$$

1316. In any Δ ABC, Δ A′B′C′ the following relationship holds :

$$\min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}$$
$$\geq 8+\max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}} = \frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2w_a}{w_b+w_c}.1.1} \overset{G-H}{\geq} \frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{6w_a}{w_b+w_c}}{\frac{2w_a}{w_b+w_c}+\frac{2w_a}{w_b+w_c}+1}$$

$$= \frac{6}{\sqrt[3]{2}}\sum_{cyc}\frac{w_a^2}{4w_a^2+w_aw_b+w_aw_c} \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \sum_{cyc}w_aw_b\leq\sum_{cyc}w_a^2}}{\geq} \frac{1}{\sqrt[3]{2}}.\frac{6\left(\sum_{cyc}w_a\right)^2}{4\sum_{cyc}w_a^2+2\sum_{cyc}w_a^2}$$

$$\overset{w_a\geq h_a=\frac{2rs}{a}\text{ and analogs}}{\geq} \frac{1}{\sqrt[3]{2}}.\frac{\left(2rs.\sum_{cyc}\frac{1}{a}\right)^2}{\sum_{cyc}s(s-a)} \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Mitrinovic}}}{\geq} \frac{1}{\sqrt[3]{2}}.\frac{4\left(2rs.\frac{9}{2s}\right)^2}{27R^2}$$

$$\Rightarrow \sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}} \overset{(\blacksquare)}{\geq} \frac{1}{\sqrt[3]{2}}.\frac{12r^2}{R^2} \overset{?}{\geq} \frac{3}{\sqrt[3]{2}}-\frac{2(R^2-4r^2)}{r^2}$$

$$\Leftrightarrow \frac{2(R^2-4r^2)}{r^2} \overset{?}{\geq} \frac{3}{\sqrt[3]{2}}\left(1-\frac{4r^2}{R^2}\right) = \frac{3}{\sqrt[3]{2}}.\frac{R^2-4r^2}{R^2}, \text{ and proving it will be complete}$$

if we can show : $2.\sqrt[3]{2}.R^2 > 3r^2 \left(\because R^2-4r^2 \overset{\text{Euler}}{\geq} 0\right) \to \text{true}$

$$\because 2.\sqrt[3]{2}.R^2 \overset{\text{Euler}}{\geq} 8.\sqrt[3]{2}.r^2 > 3r^2 \therefore \sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}} \overset{(\bullet)}{\geq} \frac{3}{\sqrt[3]{2}}-\frac{2(R^2-4r^2)}{r^2}$$

$$\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}} = \frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2m_a}{m_b+m_c}.1.1} \overset{G-H}{\geq} \frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{6m_a}{m_b+m_c}}{\frac{2m_a}{m_b+m_c}+\frac{2m_a}{m_b+m_c}+1}$$

$$=\frac{6}{\sqrt[3]{2}}\sum_{cyc}\frac{m_a^2}{4m_a^2+m_am_b+m_am_c}\overset{\substack{\text{Bergstrom}\\ and\\ \sum_{cyc}m_am_b\le\sum_{cyc}m_a^2}}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{6\left(\sum_{cyc}m_a\right)^2}{4\sum_{cyc}m_a^2+2\sum_{cyc}m_a^2}$$

$$\overset{m_a\ge h_a=\frac{2rs}{a}\ and\ analogs}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{\left(2rs.\sum_{cyc}\frac{1}{a}\right)^2}{\frac{3}{4}\sum_{cyc}a^2}\overset{\substack{\text{Bergstrom}\\ and\\ \text{Leibnitz}}}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{4\left(2rs.\frac{9}{2s}\right)^2}{27R^2}$$

$$\Rightarrow\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\overset{(\blacksquare\blacksquare)}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{12r^2}{R^2}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}-\frac{4(R-2r)}{r}$$

$$\Leftrightarrow\frac{4(R-2r)}{r}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}\cdot\frac{(R-2r)(R+2r)}{R^2}\ \textit{and proving it will be complete}$$

$$\text{if we can show}: 4R^2>3r(R+2r)\ \left(\because R^2-4r^2\overset{\text{Euler}}{\ge}0\ and\ \sqrt[3]{2}>1\right)$$

$$\Leftrightarrow(R-2r)(4R+5r)+4r^2>0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\therefore\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\bullet\bullet)}{\ge}\frac{3}{\sqrt[3]{2}}-\frac{4(R-2r)}{r}$$

$$\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2w_a}{w_b+w_c}.1.1}\overset{A-G}{\le}\frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{2w_a}{w_b+w_c}+2}{3}$$

$$=\frac{2}{3}\cdot\frac{1}{\sqrt[3]{2}}\left(\sum_{cyc}w_a\right)\left(\sum_{cyc}\frac{1}{w_b+w_c}\right)\overset{A-G}{\le}\frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}(\sqrt{3}s)\left(\sum_{cyc}\frac{1}{\sqrt{w_bw_c}}\right)\overset{\substack{w_a\ge h_a\ and\ analogs+\text{CBS}\\ and\\ \text{Mitrinovic}}}{\le}$$

$$\frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}.\sqrt{3}.\frac{3\sqrt{3}R}{2}.\sqrt{\sum_{cyc}\frac{1}{h_a}}.\sqrt{\sum_{cyc}\frac{1}{h_a}}\Rightarrow\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\blacksquare\blacksquare\blacksquare)}{\le}$$

$$\frac{3}{2.\sqrt[3]{2}}\cdot\frac{R}{r}\overset{?}{\le}\frac{3}{\sqrt[3]{2}}+\frac{4(R-2r)}{r}\Leftrightarrow\frac{4(R-2r)}{r}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}\cdot\frac{R-2r}{2r}\ \textit{and proving it will be}$$

$$\text{complete if we can show}: 8.\sqrt[3]{2}>3\ \left(\because R-2r\overset{\text{Euler}}{\ge}0\ and\ \sqrt[3]{2}>1\right)\to\text{true}$$

$$\therefore\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\bullet\bullet\bullet)}{\le}\frac{3}{\sqrt[3]{2}}+\frac{4(R-2r)}{r}$$

$$\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2m_a}{m_b+m_c}.1.1}\overset{A-G}{\le}\frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{2m_a}{m_b+m_c}+2}{3}$$

$$=\frac{2}{3}\cdot\frac{1}{\sqrt[3]{2}}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)\overset{A-G}{\le}\frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}(4R+r)\left(\sum_{cyc}\frac{1}{\sqrt{m_bm_c}}\right)$$

$$\overset{\substack{m_a \geq h_a \text{ and analogs} + \text{CBS} \\ \text{and} \\ \text{Euler}}}{\leq} \frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}\left(\frac{9R}{2}\right)\cdot\sqrt{\sum_{cyc}\frac{1}{h_a}}\cdot\sqrt{\sum_{cyc}\frac{1}{h_a}}$$

$$\Rightarrow \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}} \overset{(■■■■)}{\leq} \frac{3}{2.\sqrt[3]{2}}\cdot\frac{R}{r} \overset{?}{\leq} \frac{3}{\sqrt[3]{2}} + \frac{2(R^2-4r^2)}{r^2}$$

$$\Leftrightarrow \frac{2(R^2-4r^2)}{r^2} \overset{?}{\geq} \frac{3}{\sqrt[3]{2}}\cdot\frac{R-2r}{2r} \text{ and proving it will be complete if we can show :}$$

$$4(R+2r) > 3r\left(\because R-2r \overset{\text{Euler}}{\geq} 0 \text{ and } \sqrt[3]{2} > 1\right) \Leftrightarrow 4R+5r > 0 \to \text{true}$$

$$\therefore \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}} \overset{(\bullet\bullet\bullet)}{\leq} \frac{3}{\sqrt[3]{2}} + \frac{2(R^2-4r^2)}{r^2}$$

$$\text{Again}, \frac{R^2R'}{r^2r'} = \frac{(R^2-4r^2+4r^2)(R'-2r'+2r')}{r^2r'}$$

$$= \frac{2r'(R^2-4r^2)+4r^2(R'-2r')+8r^2r'+(R^2-4r^2)(R'-2r')}{r^2r'}$$

$$= 8 + \frac{2(R^2-4r^2)}{r^2} + \frac{4(R'-2r')}{r'} + \frac{(R^2-4r^2)(R'-2r')}{r^2r'}$$

$$\overset{\text{Euler}}{\geq} 8 + \frac{2(R^2-4r^2)}{r^2} + \frac{4(R'-2r')}{r'} \therefore \frac{R^2R'}{r^2r'} \overset{(?)}{\geq} 8 + \frac{2(R^2-4r^2)}{r^2} + \frac{4(R'-2r')}{r'}$$

$$\text{Let } m = \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}, \sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\} \text{ and}$$

$$M = max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}}, \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$$\boxed{Case\ 1}\ m = \sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}; M = \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}} \text{ and then, via } (■), (■■■■),$$

$$LHS - RHS \geq \frac{1}{\sqrt[3]{2}}\cdot\frac{12r^2}{R^2} - \frac{3}{2.\sqrt[3]{2}}\cdot\frac{R}{r} + \frac{R^2R'}{r^2r'} - 8 \overset{\text{Euler}}{\geq} -\frac{3}{\sqrt[3]{2}}\left(\frac{R}{2r} - \frac{4r^2}{R^2}\right)$$

$$+\frac{2(R^2-4r^2)}{r^2} \overset{?}{\geq} 0 \Leftrightarrow \frac{2(R^2-4r^2)}{r^2} \overset{?}{\geq} \frac{3}{\sqrt[3]{2}}\cdot\frac{R^3-8r^3}{2R^2r} \text{ and proving it will be complete}$$

$$\text{if we can show : } 4R^2(R+2r) > 3r(R^2+4r^2-2Rr)$$

$$\left(\because R-2r \overset{\text{Euler}}{\geq} 0 \text{ and } \sqrt[3]{2} > 1\right) \Leftrightarrow 4R^3+5R^2r+6r^2(R-2r) > 0 \to \text{true}$$

$$\because R \overset{\text{Euler}}{\geq} 2r \therefore \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}, \sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\} + \frac{R^2R'}{r^2r'}$$

$$\geq 8 + max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}}, \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$$\boxed{Case\ 2}\ \mathbf{m}=\sum_{cyc}\sqrt[3]{\frac{m_a{}'}{m_b{}'+m_c{}'}};\ \mathbf{M}=\sum_{cyc}\sqrt[3]{\frac{w_a{}'}{w_b{}'+w_c{}'}}\ \textit{and then, via } (\blacksquare\blacksquare),(\blacksquare\blacksquare\blacksquare),$$

$$\text{LHS}-\text{RHS}\ge\frac{1}{\sqrt[3]{2}}\cdot\frac{12r'^2}{R'^2}-\frac{3}{2.\sqrt[3]{2}}\cdot\frac{R'}{r'}+\frac{R^2R'}{r^2r'}-8\overset{\text{Euler}}{\ge}-\frac{3}{\sqrt[3]{2}}\left(\frac{R'}{2r'}-\frac{4r'^2}{R'^2}\right)$$

$$+\frac{4(R'-2r')}{r'}\overset{?}{\ge}0\Leftrightarrow\frac{4(R'-2r')}{r'}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}\cdot\frac{R'^3-8r'^3}{2R'^2r'}\ \textit{and proving it will be}$$

$$\text{complete if we can show}:8R'^2>3\left(R'^2+4r'^2-2R'r'\right)$$

$$\left(\because R'-2r'\overset{\text{Euler}}{\ge}0\ \textit{and}\ \sqrt[3]{2}>1\right)\Leftrightarrow 5R'^2+6r'(R'-2r')>0\to\text{true}$$

$$\because R'\overset{\text{Euler}}{\ge}2r'\ \therefore\ \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a{}'}{m_b{}'+m_c{}'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\ge 8+max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a{}'}{w_b{}'+w_c{}'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$$\boxed{Case\ 3}\ \mathbf{m}=\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}};\ \mathbf{M}=\sum_{cyc}\sqrt[3]{\frac{w_a{}'}{w_b{}'+w_c{}'}},\ \text{then, via } (\bullet),(\bullet\bullet\bullet),(⍰),$$

$$\text{LHS}-\text{RHS}\ge\frac{3}{\sqrt[3]{2}}-\frac{2(R^2-4r^2)}{r^2}+8+\frac{2(R^2-4r^2)}{r^2}+\frac{4(R'-2r')}{r'}-8-\frac{3}{\sqrt[3]{2}}$$

$$-\frac{4(R'-2r')}{r'}=0\ \therefore\ \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a{}'}{m_b{}'+m_c{}'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\ge 8+max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a{}'}{w_b{}'+w_c{}'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$$\boxed{Case\ 4}\ \mathbf{m}=\sum_{cyc}\sqrt[3]{\frac{m_a{}'}{m_b{}'+m_c{}'}};\ \mathbf{M}=\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}},\ \text{then, via } (\bullet\bullet),(\bullet\bullet\bullet\bullet),(⍰),$$

$$\text{LHS}-\text{RHS}\ge\frac{3}{\sqrt[3]{2}}-\frac{4(R'-2r')}{r'}+8+\frac{2(R^2-4r^2)}{r^2}+\frac{4(R'-2r')}{r'}-8-\frac{3}{\sqrt[3]{2}}$$

$$-\frac{2(R'^2-4r'^2)}{r'^2}=0\ \therefore\ \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a{}'}{m_b{}'+m_c{}'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\ge 8+max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a{}'}{w_b{}'+w_c{}'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}\ \therefore\ \text{combining } \textit{all} \text{ cases,}$$

in *any* $\Delta\,ABC,\Delta\,A'B'C'$,

$$\min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a{}'}{m_b{}'+m_c{}'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\geq 8+\max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a{}'}{w_b{}'+w_c{}'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\},''='' \text{ iff } \Delta ABC, \Delta A'B'C' \text{ are each equilateral (QED)}$$

1317. In acute ΔABC the following relationship holds:

$$\sqrt{3}\sum_{cyc}\sec A+9\sum_{cyc}\csc A\geq 24\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$f(x)=\frac{1}{\cos x}\Rightarrow f'(x)=\frac{\sin x}{\cos^2 x}>0\ (\forall)x\in\left[0;\frac{\pi}{2}\right]$$

$$f''(x)=\frac{\cos^3 x+2\cos x\sin^2 x}{\cos^4 x}>0\ (\forall)x\in\left[0;\frac{\pi}{2}\right]\Rightarrow$$

$$\Rightarrow f\to \text{convex on }\left[0;\frac{\pi}{2}\right]\Rightarrow \frac{f(A)+f(B)+f(C)}{3}\overset{Jensen}{\geq} f\left(\frac{A+B+C}{3}\right)\Rightarrow$$

$$\Rightarrow \sum\sec A\geq 3\sec\frac{\pi}{3}=\frac{3}{\cos\frac{\pi}{3}}=\frac{3}{\frac{1}{2}}=6\quad (1)$$

$$g(x)=\csc x=\frac{1}{\sin x}\Rightarrow g'(x)=\frac{-\cos x}{\sin^2 x}\Rightarrow g''(x)=\frac{\sin^3 x+2\sin x\cos^3 x}{\sin^4 x}>0$$

$$\Rightarrow g(x)\to \text{convex on }\left[0;\frac{\pi}{2}\right]\Rightarrow \frac{g(A)+g(B)+g(C)}{3}\geq g\left(\frac{A+B+C}{3}\right)$$

$$\Rightarrow \sum\csc A\geq 3\frac{1}{\sin\frac{\pi}{3}}=3\cdot\frac{1}{\frac{\sqrt{3}}{2}}=\frac{6}{\sqrt{3}}=\frac{6\sqrt{3}}{3}=2\sqrt{3}\quad (2)$$

From (1) and (2) $\Rightarrow \sqrt{3}\sum\sec A+9\sum\csc A\geq 6\sqrt{3}+18\sqrt{3}=24\sqrt{3}$

Solution 2 by Marin Chirciu-Romania

Lemma: In ΔABC holds:

$$\sum\sec A\geq 6$$

$$\sum\csc A\geq 2\sqrt{3}$$

Proof.

$$\sum \sec A = \sum \frac{1}{\cos A} = \frac{p^2 + r^2 - 4R^2}{p^2 - (2R + r)^2} \overset{Gerretsen}{\geq} 6,$$

$$\sum \csc A = \sum \frac{1}{\sin A} = \frac{p^2 + r^2 + 4Rr}{2pr} \underset{Mitrinovic}{\overset{Gerretsen}{\geq}} 2\sqrt{3}.$$

Let's get back to the main problem.Using the Lemma we obtain:

$$LHS = \sqrt{3} \sum \sec A + 9 \sum \csc A \geq 24\sqrt{3} \overset{Lemma}{\geq} \sqrt{3} \cdot 6 + 9 \cdot 2\sqrt{3} = 24\sqrt{3}$$

Equality holds if and only if the triangle is equilateral.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For acute triangle ABC

$$\sqrt{3}(\sec A + \sec B + \sec C) + 9(\csc A + \csc B + \csc C)$$

$$= \sqrt{3}\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right) + 9\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)$$

$$\geq \sqrt{3}(6) + 9(2\sqrt{3}) = \sqrt{3}(24) \text{ ok}$$

Because $\cos A + \cos B + \cos C \leq \frac{3}{2}$

$$\Rightarrow 1 \leq \frac{3}{2} \cdot \frac{1}{\cos A + \cos B + \cos C} \Rightarrow 1 \leq \frac{3}{2}\left(\frac{1}{9}\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right)\right)$$

$$\Rightarrow 6 \leq \frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}$$

and $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2} \Rightarrow 1 \leq \frac{3\sqrt{3}}{2}\left(\frac{1}{\sin A + \sin B + \sin C}\right)$

$$\Rightarrow 1 \leq \frac{3\sqrt{3}}{2}\left(\frac{1}{9}\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)\right) \Rightarrow 2\sqrt{3} \leq \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}$$

Therefore it is to be true.

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\sum \sec A \geq 3\sqrt[3]{\prod \sec A} \geq 3\sqrt[3]{\sec^3 \frac{\pi}{3}} = 6$$

$$\sum \csc A \geq 3\sqrt[3]{\prod \csc A} \geq 3\sqrt[3]{\csc \frac{\pi}{3}} = 2\sqrt{3} \Rightarrow LHS = \sqrt{3} \cdot 6 + 9 \cdot 2\sqrt{3} = 24\sqrt{3}$$

Therefore $\Rightarrow \sqrt{3}\sum_{cyc} \sec A + 9\sum_{cyc} \csc A \geq 24\sqrt{3}$

1318. In any ΔABC the following relationship holds:

$$\sqrt{\frac{a^2+bc}{b^2+ac}}+\sqrt{\frac{b^2+ac}{c^2+ab}}+\sqrt{\frac{c^2+ab}{a^2+bc}}+\frac{R^2}{4r^2}\geq 1+\frac{2a}{b+c}+\frac{2b}{a+c}+\frac{2c}{a+b}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Tapas Das-India

First of all, we can easily prove the following equality

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}$$

Consequently, it is sufficient to prove

$\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}\leq\frac{R}{6r}+\frac{F}{6}$. Let $f(s^2)=\frac{R}{6r}+\frac{7}{6}-\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}$

f is a decreasing function and we need to prove $f(s^2)\geq 0$

Applying Gerretsen's inequality we deduce that: $s^2\leq 4R^2+4Rr+3r^2$

Therefore, it is sufficient to prove $f(4R^2+4Rr+3r^2)\geq 0$

After some simplification, we get

$$(R-2r)(2(R-2r)+5(R-2r)+r^2)\geq 0$$

(Euler)

$$\therefore\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{R}{6r}+\frac{7}{6}$$

We need to show: $1+2\left(\frac{R}{6r}+\frac{7}{6}\right)\leq\frac{R^2}{4r^2}+3$

or $\frac{R^2}{4r^2}+3-1-2\left(\frac{R}{6r}+\frac{7}{6}\right)\geq 0$ or $\frac{R^2}{4r^2}-\frac{R}{3r}-\frac{1}{3}\geq 0$

or $(R-2r)(3R+2r)\geq 0$ (True) Euler

Note: $\sqrt{\frac{a^2+bc}{b^2+ac}}+\sqrt{\frac{b^2+ac}{c^2+ab}}+\sqrt{\frac{c^2+ab}{a^2+bc}}\geq 3$

By (AM-GM)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$1)\ \sum\sqrt{\frac{a^2+bc}{b^2+ca}}+\frac{R^2}{4r^2}\overset{AM\geq GM}{\geq}3+\frac{R^2}{4r^2}\quad (1)$$

$$2)\ 1+2\sum\frac{a}{b+c}=1+\frac{1}{2}\sum a\cdot\frac{4}{b+c}\le 1+\frac{1}{2}\sum a\left(\frac{1}{b}+\frac{1}{c}\right)$$

$$=1+\frac{1}{2abc}\cdot\sum a^2(b+c)=1+\frac{\sum a\sum ab-3abc}{2abc}=$$

$$=1-\frac{3}{2}+\frac{\sum a\cdot\sum ab}{2abc}=-\frac{1}{2}+\frac{2p(p^2+4Rr+r^2)}{8p\cdot Rr}\le$$

$$\overset{Gerretsen}{\le}-\frac{1}{2}+\frac{4R^2+8Rr+4r^2}{4Rr}=2-\frac{1}{2}+\frac{R}{r}+\frac{r}{R}=$$

$$=\frac{3}{2}+\frac{R}{r}+\frac{r}{R}\overset{?}{\le}3+\frac{R^2}{4r^2}$$

$$\frac{R}{r}=y$$

$$\frac{y^2}{4}+\frac{3}{2}-y-\frac{1}{y}\ge 0$$

$$y^3-4y^2+6y-4\ge 0$$

$$(y-2)(y^2-2y+2)=\underbrace{(y-2)}_{\ge 0}\underbrace{((y-1)^2+1)}_{>0}\ge 0$$

1319. In any Δ ABC, the following relationship holds :

$$\min\left\{\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\right\}+\frac{R+\sqrt{3}s}{r}\ge 11+2\cdot\max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\overset{A-G}{\ge}3\ \therefore\ \min\left\{\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\right\}\ge 3$$

$$\therefore \text{it suffices to prove : }\boxed{\frac{R+\sqrt{3}s}{r}\overset{(\bullet)}{\ge}8+2.\max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\}}$$

$$\text{Now, }8+2\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{A-G}{\le}8+\frac{\sum_{cyc}a^3}{abc}=8+\frac{2s(s^2-6Rr-3r^2)}{4Rrs}$$

$$=\frac{s^2+10Rr-3r^2}{2Rr}\overset{?}{\le}\frac{R+\sqrt{3}s}{r}\Leftrightarrow\frac{R^2+3s^2+2\sqrt{3}Rs}{r^2}\underset{(*)}{\overset{?}{\ge}}\frac{(s^2+10Rr-3r^2)^2}{4R^2r^2}$$

$$\text{Now, LHS of } (*) \overset{\text{Mitrinovic}}{\geq} \frac{R^2+3s^2+2\sqrt{3}s.\frac{2s}{3\sqrt{3}}}{r^2}$$

$$= \frac{3R^2+13s^2}{3r^2} \overset{?}{\geq} \frac{(s^2+10Rr-3r^2)^2}{4R^2r^2} \Leftrightarrow 4R^2(3R^2+13s^2) \overset{?}{\geq} 3(s^2+10Rr-3r^2)^2$$

$$\Leftrightarrow 12R^4-300R^2r^2+180Rr^3-27r^4+(52R^2-60Rr+18r^2)s^2 \underset{(**)}{\overset{?}{\geq}} 3s^4$$

$$\text{Now, } 3s^4 \overset{\text{Gerretsen}}{\leq} (12R^2+12Rr+9r^2)s^2$$

$$\overset{?}{\leq} 12R^4-300R^2r^2+180Rr^3-27r^4+(52R^2-60Rr+18r^2)s^2$$

$$\Leftrightarrow (40R^2-72Rr+9r^2)s^2+12R^4-300R^2r^2+180Rr^3-27r^4 \underset{(***)}{\overset{?}{\geq}} 0$$

$$\text{Again, LHS of } (***) \overset{\text{Gerretsen}}{\geq}$$

$$(40R^2-72Rr+9r^2)(16Rr-5r^2)+12R^4-300R^2r^2+180Rr^3-27r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3t^4+160t^3-413t^2+171t-18 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(3t^3+125t^2+41t(t-2)+t+9) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \boxed{\frac{R+\sqrt{3}s}{r} \overset{(\blacksquare)}{\geq} 8+2\sum_{cyc}\frac{a^2}{b^2+c^2}}$$

$$\text{Also, } \frac{R+\sqrt{3}s}{r} \overset{\text{Mitrinovic}}{\geq} \frac{R}{r}+9 \overset{?}{\geq} 8+2\sum_{cyc}\frac{a}{b+c}$$

$$\Leftrightarrow \frac{R}{r}+1 \overset{?}{\geq} 2\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{b+c}\right)-6$$

$$\Leftrightarrow \frac{R}{r}+7 \overset{?}{\geq} \frac{4s}{2s(s^2+2Rr+r^2)}.\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)$$

$$\Leftrightarrow \frac{R+7r}{r} \overset{?}{\geq} \frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2} \Leftrightarrow (R-3r)s^2+r(2R^2+7Rr+5r^2) \underset{(****)}{\overset{?}{\geq}} 0$$

$\boxed{\textit{Case } 1}$ $R-3r \geq 0$ *and* **then : LHS of** $(****) \geq r(2R^2+7Rr+5r^2) > 0$

$\Rightarrow (****)$ **is true (strict inequality)**

$\boxed{\textit{Case } 2}$ $R-3r < 0$ *and* **then : LHS of** $(****)$

$$= -(3r-R)s^2+r(2R^2+7Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\geq} -(3r-R)(4R^2+4Rr+3r^2)+r(2R^2+7Rr+5r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2t^3-3t^2-t-2 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(2t^2+t+1) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (****) \text{ is true } \therefore \boxed{\frac{R+\sqrt{3}s}{r} \overset{(\blacksquare\blacksquare)}{\geq} 8+2\sum_{cyc}\frac{a}{b+c}}$$

$$\therefore (\blacksquare),(\blacksquare\blacksquare) \Rightarrow \frac{R+\sqrt{3}s}{r} \geq 8+2.max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\} \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \min\left\{\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\right\}+\frac{R+\sqrt{3}s}{r} \geq 11+2.max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\}$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

We show that:

$$\sum\frac{a^2}{b^2+c^2} \geq \sum\frac{a}{b+c}$$

We have, $\frac{a^2}{b^2+c^2}-\frac{a}{b+c}=\frac{ab(a-b)+ac(a-c)}{(b^2+c^2)(b+c)}$

$$\frac{b^2}{c^2+a^2}-\frac{b}{c+a}=\frac{bc(b-c)+ab(b-a)}{(c^2+a^2)(b+c)}$$

$$\frac{c^2}{a^2+b^2}-\frac{c}{a+b}=\frac{ac(c-a)+bc(c-b)}{(b^2+a^2)(b+a)}$$

Now we obtain,

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}-\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=\sum\frac{ab(a-b)+ac(b-c)}{(b^2+c^2)(b+c)}$$

$$=(a^2+b^2+c^2+ab+bc+ca)\sum\frac{ab(a-b)^2}{(b+c)(c+a)(b^2+c^2)(c^2+a^2)} \geq 0$$

$$\therefore \sum\frac{a^2}{b^2+c^2} \geq \sum\frac{a}{b+c}$$

$$\therefore \max\left\{\sum\frac{a^2}{b^2+c^2};\sum\frac{a}{b+c}\right\} \leq \sum\frac{a^2}{b^2+c^2}$$

$\sum\sqrt{\frac{r_a}{r_b}} \geq 3$ **(AM-GM),** $\sum\sqrt{\frac{w_a}{w_b}} \geq 3$ **(AM-GM)**

$$\therefore \min\left\{\sum\sqrt{\frac{r_a}{r_b}},\sum\sqrt{\frac{w_a}{w_b}}\right\} \geq 3$$

We need to show

$$3+\frac{R+\sqrt{3}s}{r}\geq 11+2\sum\frac{a^2}{b^2+c^2} \text{ or } \frac{R+\sqrt{3}s}{r}\geq 8+2\sum\frac{a^2}{b^2+c^2}$$

$$\text{now } \frac{R+\sqrt{3}s}{r}\geq\frac{R+\sqrt{3}\cdot 3\sqrt{3}r}{r}=\frac{R}{r}+9 \quad (1)$$

$$(\because s^2\geq 27r^2)$$

$$\sum\frac{a^2}{b^2+c^2}=\sum_{cyc}\left(\frac{a^2}{b^2+c^2}-\frac{1}{2}\right)+\frac{3}{2}=\sum\frac{2a^2-b^2-c^2}{2(b^2+c^2)}+\frac{3}{2}$$

$$=\sum_{cyc}\frac{a^2-b^2}{2(b^2+c^2)}+\sum_{cyc}\frac{a^2-c^2}{2(b^2+c^2)}+\frac{3}{2}$$

$$=\sum(a^2-b^2)\left(\frac{1}{2(b^2+c^2)}-\frac{1}{2(a^2+c^2)}\right)+\frac{3}{2}$$

$$=\sum\frac{(a-b)^2(a+b)^2}{2(a^2+c^2)(b^2+c^2)}+\frac{3}{2}$$

According Cauchy – Schwarz,

$$(a^2+c^2)(c^2+b^2)\geq c^2(a+b)^2$$

By AM-GM

$$c^2=\frac{(c+a-b+c+b-a)^2}{4}\geq(c+a-b)(c+b-a)$$

$$\therefore (a^2+c^2)(c^2+b^2)\geq(c+a-b)(c+b-a)(a+b)^2$$

$$\therefore \sum\frac{a^2}{b^2+c^2}=\sum\frac{(a-b)^2(a+b)^2}{2(a^2+c^2)(b^2+c^2)}+\frac{3}{2}\leq\sum\frac{(a-b)^2}{2(c+a-b)(c+a-b)}+\frac{3}{2}$$

$$=\frac{\sum(a+b-c)\,(a-b)^2}{2(a+b-c)(b+c-a)(c+a-b)}+\frac{3}{2}=\frac{abc}{(a+b-c)(b+c-a)(c+a-b)}+\frac{1}{2}$$

$$=\frac{R}{2r}+\frac{1}{2}$$

$$\therefore 2\sum\frac{a^2}{b^2+c^2}\leq\frac{R}{r}+1$$

$$\therefore 8+2\sum\frac{a^2}{b^2+c^2}\leq 8+\frac{R}{r}+1=9+\frac{R}{r}\leq\frac{R+\sqrt{3}s}{r}$$

(using (1))

1320. In ΔABC the following relationship holds:

$$5-\frac{4r}{R}\le\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\le\frac{2R}{r}-1$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=\frac{a^3+b^3+c^3}{abc}=\frac{2(s^3-3r^2S-6Rrs)}{4Rrs}=\frac{s^2-3r^2-6Rr}{2Rr}$$

$$\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2-3r^2-6Rr}{2Rr}=\frac{4R^2+4Rr-6Rr}{2Rr}=\frac{4R^2-2Rr}{2Rr}$$

$$=\frac{2R}{r}-1$$

Again,

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=\frac{s^2-3r^2-6Rr}{2Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2-3r^2-6Rr}{2Rr}$$

$$=\frac{10Rr-8r^2}{2Rr}=5-\frac{4r}{R}$$

1321. In any Δ ABC, and $\forall$ n $\ge$ 2, the following relationship holds :

$$\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}+\left(\frac{R}{2r}\right)^n\ge 1+\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z\therefore s=x+y+z\Rightarrow a=y+z,$$
$$b=z+x \text{ and } c=x+y$$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(1)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\text{ and }1+\frac{4R}{r}$$
$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$
$$\Rightarrow 1+\frac{4R}{r}\overset{(2)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, }\sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z}\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(3)}{=}\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\therefore(1),(2),(3)\Rightarrow\frac{s^2}{r^2}$$

$$\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

Now $\forall$ u, v, w > 0, $u^3+u^3+v^3\overset{A-G}{\geq}3u^2v, v^3+v^3+w^3\overset{A-G}{\geq}3v^2w$ *and* $w^3+w^3+u^3\overset{A-G}{\geq}3w^2u$ ∴ summing up : $\sum_{cyc}u^3\geq\sum_{cyc}u^2v$ *and* choosing $u=xy$,

$$v=yz\ and\ w=zx, \sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}xy^2\right)\ and\ \sum_{cyc}x^4y^2\geq 3x^2y^2z^2$$

$$\therefore\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2\Rightarrow (i)\text{ is true}$$

$$\Rightarrow\frac{s^2}{r^2}\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(\blacklozenge)}{\leq}\frac{s^2}{r(4R+r)}$$

Also, $\sum_{cyc}\frac{a}{b}=\sum_{cyc}\frac{y+z}{z+x}\Rightarrow\sum_{cyc}\frac{a}{b}\overset{(4)}{=}\frac{\sum_{cyc}(x+y)(y+z)^2}{\prod_{cyc}(y+z)}\therefore (1),(2),(4)\Rightarrow\frac{s^2}{r^2}$

$$\geq\left(\sum_{cyc}\frac{a}{b}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)(y+z)^2}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)(y+z)^2\right)$$

$$\Leftrightarrow\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\overset{(ii)}{\geq}xyz\left(\sum_{cyc}x^2y\right)+3x^2y^2z^2$$

Now $\forall$ u, v, w > 0, $v^3+v^3+u^3\overset{A-G}{\geq}3v^2u, w^3+w^3+v^3\overset{A-G}{\geq}3w^2v$ *and* $u^3+u^3+w^3\overset{A-G}{\geq}3u^2w$ ∴ summing up : $\sum_{cyc}u^3\geq\sum_{cyc}uv^2$ *and* choosing $u=xy$,

$$v=yz\ and\ w=zx, \sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}x^2y\right) and\ \sum_{cyc}x^2y^4\geq 3x^2y^2z^2$$

$$\therefore\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}x^2y\right)+3x^2y^2z^2\Rightarrow (ii)\text{ is true}$$

$$\Rightarrow \frac{s^2}{r^2} \ge \left(\sum_{cyc}\frac{a}{b}\right)\left(1+\frac{4R}{r}\right) \Rightarrow \sum_{cyc}\frac{a}{b} \overset{(\blacklozenge\blacklozenge)}{\le} \frac{s^2}{r(4R+r)}$$

$$\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}} = \frac{1}{\sqrt[3]{5}}\cdot\sum_{cyc}\sqrt[3]{\frac{5a}{2b+3c}\cdot 1.1} \overset{G-H}{\ge} \frac{1}{\sqrt[3]{5}}\cdot\sum_{cyc}\frac{\frac{15a}{2b+3c}}{\frac{5a}{2b+3c}+\frac{5a}{2b+3c}+1}$$

$$= \frac{15}{\sqrt[3]{5}}\cdot\sum_{cyc}\frac{a^2}{10a^2+2ab+3ca} \overset{Bergstrom}{\ge} \frac{1}{\sqrt[3]{5}}\cdot\frac{15\left(\sum_{cyc}a\right)^2}{10\sum_{cyc}a^2+5\sum_{cyc}ab}$$

$$= \frac{1}{\sqrt[3]{5}}\cdot\frac{15\left(\sum_{cyc}a\right)^2}{10\sum_{cyc}a^2+5\sum_{cyc}a^2} = \frac{1}{\sqrt[3]{5}}\cdot\frac{4s^2}{\sum_{cyc}a^2} \Rightarrow \boxed{\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}} \overset{(\bullet)}{\ge} \frac{1}{\sqrt[3]{5}}\cdot\frac{2s^2}{s^2-4Rr-r^2}}$$

$$\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}} = \frac{1}{\sqrt[3]{5}}\cdot\sum_{cyc}\sqrt[3]{\frac{5a}{3b+2c}\cdot 1.1} \overset{A-G}{\le} \frac{1}{\sqrt[3]{5}}\cdot\sum_{cyc}\frac{\frac{5a}{3b+2c}+2}{3}$$

$$= \frac{1}{\sqrt[3]{5}}\cdot\left(\frac{5}{3}\sum_{cyc}\frac{a}{3b+2c}+2\right) \overset{Weighted\ A-H}{\le} \frac{1}{\sqrt[3]{5}}\cdot\left(\frac{5}{3}\sum_{cyc}\frac{a}{\left(\frac{25}{\frac{3}{b}+\frac{2}{c}}\right)}+2\right)$$

$$= \frac{1}{\sqrt[3]{5}}\cdot\left(\frac{1}{15}\sum_{cyc}\frac{3ca+2ab}{bc}+2\right) = \frac{1}{\sqrt[3]{5}}\cdot\left(\frac{1}{5}\sum_{cyc}\frac{a}{b}+\frac{2}{15}\sum_{cyc}\frac{a}{c}+2\right) \overset{via\ (\blacklozenge),(\blacklozenge\blacklozenge)}{\le}$$

$$\frac{1}{\sqrt[3]{5}}\cdot\left(\frac{s^2}{3r(4R+r)}+2\right) \Rightarrow \boxed{\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}} \overset{(\bullet\bullet)}{\le} \frac{1}{\sqrt[3]{5}}\cdot\left(\frac{s^2+24Rr+6r^2}{3r(4R+r)}\right)}$$

$$\therefore \sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}+\frac{R^2-4r^2}{4r^2}-\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}} \overset{via\ (\bullet).(\bullet\bullet)}{\ge}$$

$$\frac{1}{\sqrt[3]{5}}\cdot\left(\frac{2s^2}{s^2-4Rr-r^2}-\frac{s^2+24Rr+6r^2}{3r(4R+r)}\right)+\frac{R^2-4r^2}{4r^2} \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{R^2-4r^2}{4r^2} \overset{?}{\ge} \frac{1}{\sqrt[3]{5}}\cdot\left(\frac{s^2+24Rr+6r^2}{3r(4R+r)}-\frac{2s^2}{s^2-4Rr-r^2}\right)$$

$$\Leftrightarrow \frac{R^2-4r^2}{4r^2} \underset{(*)}{\overset{?}{\ge}} \frac{1}{\sqrt[3]{5}}\cdot\frac{(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2}{3r(4R+r)(s^2-4Rr-r^2)}$$

Now, $(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2$

$$= s^4-(4Rr+r^2)s^2-6(4Rr+r^2)^2 \overset{Gerretsen+Euler}{\ge} 3(4Rr+r^2)s^2-(4Rr+r^2)s^2$$

$$-6(4Rr+r^2)^2 = 2(4Rr+r^2)(s^2-12Rr-3r^2) \overset{Gerretsen+Euler}{\ge} 0$$

$$\therefore \frac{1}{\sqrt[3]{5}}\cdot\frac{(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2}{3r(4R+r)(s^2-4Rr-r^2)}$$

$$\overset{\sqrt[3]{5}>\frac{5}{3}}{\leq}\frac{3}{5}\cdot\frac{(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2}{3r(4R+r)(s^2-4Rr-r^2)}\overset{?}{\leq}\frac{R^2-4r^2}{4r^2}$$

$$\Leftrightarrow -4rs^4+(20R^3+5R^2r-64Rr^2-16r^3)s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\underset{(**)}{\overset{?}{\geq}}0$$

$$\text{Now, LHS of } (**)\overset{\text{Gerretsen}}{\geq}\begin{pmatrix}-4r(4R^2+4Rr+3r^2)+20R^3\\+5R^2r-64Rr^2-16r^3\end{pmatrix}s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow (20R^3-11R^2r-80Rr^2-28r^3)s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\underset{(***)}{\overset{?}{\geq}}0$$

Case 1 $20R^3-11R^2r-80Rr^2-28r^3\geq 0$ *and* then, LHS of $(***)$

$$\overset{\text{Gerretsen}}{\geq}(20R^3-11R^2r-80Rr^2-28r^3)(16Rr-5r^2)$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow 120t^4-158t^3-263t^2+152t+92\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(120t^2+322t+545)+1044\right)$$

$$\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow (***)\text{ is true}$$

Case 2 $20R^3-11R^2r-80Rr^2-28r^3<0$ *and* then, LHS of $(***)$

$$=-\left(-(20R^3-11R^2r-80Rr^2-28r^3)\right)s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)$$

$$\overset{\text{Gerretsen}}{\geq}-\left(-(20R^3-11R^2r-80Rr^2-28r^3)\right)(4R^2+4Rr+3r^2)$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow 40t^5-22t^4-172t^3+117t^2-20\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)(40t^4+30t^3+28t^2(t-2)+5t+10)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow (***)$ is true $\therefore$ combining *cases* 1 *and* 2, $(***)\Rightarrow(**)\Rightarrow(*)$ is true $\forall\,\Delta\,ABC$

$$\therefore\frac{R^2-4r^2}{4r^2}\overset{(\blacksquare)}{\geq}\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}-\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}\text{ and }(n-2).\ln\left(\frac{R}{2r}\right)\overset{\text{Euler}}{\geq}0$$

$$\Rightarrow \ln\left(\frac{R}{2r}\right)^n\geq\ln\left(\frac{R}{2r}\right)^2\Rightarrow\left(\frac{R}{2r}\right)^n-1\geq\frac{R^2-4r^2}{4r^2}\overset{\text{via }(\blacksquare)}{\geq}$$

$$\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}-\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}\Rightarrow\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}+\left(\frac{R}{2r}\right)^n$$

$\geq 1+\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}$ in *any* Δ ABC *and* $\forall\, n\geq 2$, " = " iff Δ ABC is equi*lateral* (QED)

1322. In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^n+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^n\ge 2, n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Adrian Popa – Romania

$$\frac{w_aw_bw_c}{r_ar_br_c}\ge\frac{h_ah_bh_c}{r_ar_br_c}=\frac{\frac{2s}{a}\cdot\frac{2s}{b}\cdot\frac{2s}{c}}{\frac{s}{p-a}\cdot\frac{s}{p-b}\cdot\frac{s}{p-c}}=$$

$$=\frac{8(s-a)(s-b)(s-c)}{abc}=\frac{8p(s-a)(s-b)(s-c)}{p\cdot abc}=$$

$$=\frac{8s^2}{p\cdot 4Rs}=\frac{8\cdot r\cdot p}{p\cdot 4R}=\frac{2r}{R}\Rightarrow$$

$$\Rightarrow\left(\frac{R}{2r}\right)^n+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^n\ge\left(\frac{R}{2r}\right)^n+\left(\frac{2r}{R}\right)^n\overset{MA\ge MG}{\ge}2\sqrt{\left(\frac{R}{2r}\right)^n\cdot\left(\frac{2r}{R}\right)^n}=2$$

Solution 2 by Tapas Das – India

$$\frac{R}{2r}=\frac{abc}{(a+b-c)(b+c-a)(c+a-b)}=\frac{abc}{8(s-a)(s-b)(s-c)}$$

$$\frac{w_aw_bw_c}{r_ar_br_c}\ge\frac{h_ah_bh_c}{r_ar_br_c}=\frac{8F^3}{abcs^2r}=\frac{8F\cdot s(s-a)(s-b)(s-c)}{abc\cdot s^2\cdot r}$$

$$=\frac{8F(s-a)(s-b)(s-c)}{abc\cdot F}=\frac{8(s-a)(s-b)(s-c)}{abc}$$

$$\therefore\left(\frac{R}{2r}\right)^n+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^n\ge$$

$$\ge\left[\frac{abc}{8(s-a)(s-b)(s-c)}\right]^2+\left[\frac{8(s-a)(s-b)(s-c)}{abc}\right]^n$$

$$\overset{AM-GM}{\ge}2\sqrt{\left[\frac{abc}{8(s-a)(s-b)(s-c)}\cdot\frac{8(s-a)(s-b)(s-c)}{abc}\right]^n}=2$$

1323. In any Δ ABC, the following relationships holds :

$$\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{n_a}{n_b}}\right)\geq\sum_{cyc}\sqrt{\frac{n_b}{n_a}};\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{g_a}{g_b}}\right)\geq\sum_{cyc}\frac{g_b}{g_a};$$

and in acute Δ ABC with $a=\min\{a,b,c\}$, holds : $g_a+h_a>n_a$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Tapas Das-India

$$\frac{n_a}{n_b}+\frac{n_b}{n_c}+\frac{n_c}{n_a}\leq\sqrt{\left(\sum n_a^2\right)\left(\sum\frac{1}{n_a^2}\right)}\quad (1)$$

Since $n_a\geq h_a$ (analog)

$$\therefore\frac{1}{n_a^2}+\frac{1}{n_b^2}+\frac{1}{n_c^2}\leq\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}=\frac{\sum a^2}{(2F)^2}\overset{Leibnitz's}{\leq}\left(\frac{3R}{2F}\right)^2$$

$$n_a^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-s\cdot\frac{a^2-(b-c)^2}{a}$$

$$=s^2-\frac{4s(s-b)(s-c)}{a}=s^2-\frac{4s^2r^2}{a(s-a)}$$

$$=s^2-4sr^2\left(\frac{1}{s-a}+\frac{1}{a}\right)=s^2-4rr_a-2rh_a\text{ (analog)}$$

$$\therefore\sum n_a^2=3s^2-4r\left(\sum r_a\right)-2r\left(\sum h_a\right)$$

$$=3s^2-4r(4R+r)-2r\frac{s^2+r(4R+r)}{2R}=\frac{(3R-r)s^2-r(4R+r)^2}{R}$$

By Doucets, $(4R+r)^2\geq 3s^2$ we get

$$\sum n_a^2\leq\frac{(3R-r)s^2-3rs^2}{R}=\left(3-\frac{4r}{R}\right)s^2\overset{AM-GM}{\leq}\left(\frac{R}{2r}\right)^2\cdot s^2=\left(\frac{Rs}{2r}\right)^2$$

$$\therefore\sum\frac{n_a}{n_b}\leq\frac{Rs}{2r}\cdot\frac{3R}{2F}=3\left(\frac{R}{2r}\right)^2$$

$$\text{Now }\sqrt{\frac{n_a}{n_b}}\overset{CBS}{\leq}\sqrt{3\left(\sum\frac{n_a}{n_b}\right)}\leq\sqrt{3\cdot 3\cdot\left(\frac{R}{2r}\right)^2}=\frac{3R}{2r}$$

$$\therefore\frac{R}{2r}\left(\sum\sqrt{\frac{n_a}{n_b}}\right)\geq\frac{3R}{2r}\text{ (AM-GM) [Since }\sqrt{\frac{n_a}{n_b}}\geq 3]$$

$$\therefore \frac{R}{2r}\left(\sum\sqrt{\frac{n_a}{n_b}}\right) \geq \sqrt{\frac{n_a}{n_b}}$$

$$\sum\frac{g_b}{g_a} \leq \sqrt{\left(\sum g_b^2\right)\left(\sum\frac{1}{g_a^2}\right)} \leq \sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{r_a^2}\right)}$$

$$= \sqrt{s^2 \cdot \frac{\sum a^2}{4F^2}} \overset{Leibnitz}{\leq} \sqrt{\frac{9R^2}{4r^2}}$$

Now $\frac{R^2}{4r^2}\left(\sum\sqrt{\frac{g_a}{g_b}}\right) \geq \frac{R^2}{4r^2} \cdot 3$ (AM-GM)

We need to show

$\frac{3R^2}{4r^2} \geq \frac{3R}{2r}$ or $\frac{R}{2r} \geq 1 \therefore R \geq 2r$ (Euler)

Note: $9a^2 = s(s-a) - \frac{(b-c)^2(s-a)}{a} \leq s(s-a)$ (analog)

If we take: $a = c = 6$ and $b = 2$

We have $s = \frac{6+6+2}{2} = 7$

$$n_a^2 = \frac{b^2(s-c) + c^2(s-b) - a(s-c)(s-b)}{a}$$

$$= \frac{4(7-6)+36(7-2)-6(7-6)(7-2)}{6} = \frac{77}{3} \therefore n_a = \sqrt{\frac{77}{3}} = 5 \cdot 06 \text{ (APP)}$$

$$h_a^2 = \frac{4F^2}{a^2} = \frac{4s(s-a)(s-b)(s-c)}{a^2} = \frac{4 \times 7 \times 1 \times 1 \times 5}{9} = \frac{35}{9}$$

$$\therefore h_a = \sqrt{\frac{35}{9}} = 1 \cdot 97 \text{ (APP)}$$

$$g_a^2 = s(s-a) - \frac{(s-a)(b-c)^2}{a}$$

$$= 7 - \frac{16}{6} = \frac{13}{3} \therefore g_a = \sqrt{\frac{13}{3}} = 2 \cdot 08$$

$$\therefore g_a + h_a = 2.08 + 1.97 = 4.05 < n_a$$

$$(\because n_a = 5.06)$$

$\therefore$ so 3rd problem-Not always true

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2 + a(s-b)(s-c)$$

$$\text{and } b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2 + a(s-b)(s-c) \text{ and } (m)+(n) \Rightarrow$$

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$

$$= 2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) + a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2$$

$$= 2\left(n_a^2+g_a^2\right) \Rightarrow 4m_a^2 + (b-c)^2 = 2\left(n_a^2+g_a^2\right) \Rightarrow (b-c)^2 + 4s(s-a) + (b-c)^2$$

$$= 2\left(n_a^2+g_a^2\right) \Rightarrow n_a^2 + g_a^2 \overset{(1)}{=} (b-c)^2 + 2s(s-a)$$

$$\text{Now, } \sum_{cyc}\frac{n_a}{R} = \sum_{cyc}\frac{2n_a h_a}{bc} \le \sum_{cyc}\frac{2n_a g_a}{bc} \overset{A-G}{\le} \sum_{cyc}\frac{n_a^2+g_a^2}{bc} \overset{\text{via }(1)}{=} \frac{(b-c)^2+2s(s-a)}{bc}$$

$$= \frac{1}{4Rrs}\left(\sum_{cyc}\left(a(b^2+c^2-2bc)\right) + 2s\sum_{cyc}\left(a(s-a)\right)\right)$$

$$= \frac{1}{4Rrs}\left(2s(s^2+4Rr+r^2) - 36Rrs + 2s\left(2s^2 - 2(s^2-4Rr-r^2)\right)\right)$$

$$= \frac{s^2-6Rr+3r^2}{2Rr} \Rightarrow 2r\sum_{cyc} n_a \overset{(*)}{\le} s^2 - 6Rr + 3r^2$$

$$\sum_{cyc}\sqrt{\frac{n_b}{n_a}} \overset{CBS}{\le} \sqrt{\sum_{cyc} n_a}\cdot\sqrt{\sum_{cyc}\frac{1}{n_a}} \le \sqrt{\sum_{cyc} n_a}\cdot\sqrt{\sum_{cyc}\frac{1}{h_a}} = \sqrt{\frac{\sum_{cyc} n_a}{r}} \overset{?}{\le} \frac{3R}{2r}$$

$$\Leftrightarrow \frac{\sum_{cyc} n_a}{r} \overset{?}{\le} \frac{9R^2}{4r^2} \Leftrightarrow 2r\sum_{cyc} n_a \underset{(\bullet)}{\overset{?}{\le}} \frac{9R^2}{2} \text{ and via } (*),$$

$$2r\sum_{cyc} n_a \le s^2 - 6Rr + 3r^2 \overset{?}{\le} \frac{9R^2}{2} \Leftrightarrow 2s^2 \underset{(\bullet\bullet)}{\overset{?}{\le}} 9R^2 + 12Rr - 6r^2 \text{ and via Gerretsen,}$$

$$2s^2 \overset{\text{Gerretsen}}{\le} 8R^2 + 8Rr + 6r^2 \overset{?}{\le} 9R^2 + 12Rr - 6r^2 \Leftrightarrow R^2 + 4Rr - 12r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R-2r)(R+6r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\Rightarrow \sum_{cyc}\sqrt{\frac{n_b}{n_a}} \le \frac{3R}{2r} \overset{A-G}{\le} \frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{n_a}{n_b}}\right) \therefore \boxed{\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{n_a}{n_b}}\right) \ge \sum_{cyc}\sqrt{\frac{n_b}{n_a}}}$$

$$\text{Again, } AI^2 = bc - 4Rr \Leftrightarrow \left(\frac{r}{\left(\frac{r}{4R}\right)}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2$$

$$= 16R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 16R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow \sin\frac{B}{2}\sin\frac{C}{2}$$

$$= \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{A}{2} \Leftrightarrow \cos\frac{B+C}{2} = \sin\frac{A}{2} \to \text{true}$$

$$\therefore AI^2 = bc - 4Rr \text{ and analogs } \therefore \sum_{cyc}\frac{g_b}{g_a} \overset{\text{Triangle-inequality}}{\le} \sum_{cyc}\frac{AI+r}{h_a}$$

$$=\sum_{cyc}\frac{a.AI+r.a}{2rs}\overset{CBS}{\leq}\frac{1}{2rs}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}AI^2}+\frac{2rs}{2rs}\overset{Leibnitz}{\leq}\frac{3R}{2rs}.\sqrt{\sum_{cyc}bc-12Rr}+1$$

$$=\frac{3R.\sqrt{s^2-8Rr+r^2}}{2rs}+1\overset{?}{\leq}\frac{3R^2}{4r^2}\Leftrightarrow\frac{s^2(3R^2-4r^2)^2}{4r^2}\overset{?}{\geq}9R^2(s^2-8Rr+r^2)$$

$$\Leftrightarrow(9R^4-60R^2r^2+16r^4)s^2+r^3(288R^3-36R^2r)\underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}}0$$

$\boxed{Case\ 1}$ $9R^4-60R^2r^2+16r^4\geq 0$ and then, LHS of $(\bullet\bullet\bullet)$

$\geq 270R^3+18R^2(R-2r)\overset{Euler}{\geq}270R^3>0\Rightarrow(\bullet\bullet\bullet)$ is true (strict inequality)

$\boxed{Case\ 2}$ $9R^4-60R^2r^2+16r^4<0$ and then, LHS of $(\bullet\bullet\bullet)$

$$=-s^2.\left(-(9R^4-60R^2r^2+16r^4)\right)+r^3(288R^3-36R^2r)\overset{Gerretsen}{\geq}$$

$$-(4R^2+4Rr+3r^2).\left(-(9R^4-60R^2r^2+16r^4)\right)+r^3(288R^3-36R^2r)\overset{?}{\geq}0$$

$$\Leftrightarrow 36t^6+36t^5-213t^4+48t^3-152t^2+64t+48\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(36t^5+108t^4+3t^3+26t^2+22t(t-2)+6(t-2)(t+2)\right)\overset{?}{\geq}0\to\text{true}$$

$\because t\overset{Euler}{\geq}2\Rightarrow(\bullet\bullet\bullet)$ is true and combining cases 1 and 2, $(\bullet\bullet\bullet)$ is true $\forall$ triangles

$$\Rightarrow\sum_{cyc}\frac{g_b}{g_a}\leq\frac{3R^2}{4r^2}\overset{A-G}{\leq}\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{g_a}{g_b}}\right)\therefore\boxed{\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{g_a}{g_b}}\right)\geq\sum_{cyc}\frac{g_b}{g_a}}$$

We now *shall* prove : $g_a+h_a>n_a$ *in acute* Δ ABC with $a=\min\{a,b,c\}$

$$\frac{b+c}{a}\geq\frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

$$\Leftrightarrow(b+c)(a+b-c).(b+c-a)(c+a-b)\geq 2a^2bc$$

$$\Leftrightarrow(ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)\geq 2a^2bc$$

$$\Leftrightarrow 2a^2b^2+2a^2bc+2ab(b^2-c^2)-(a^2+b^2-c^2)(ab+ac+b^2-c^2)\geq 2a^2bc$$

$$\Leftrightarrow 2a^2b^2-(a^2+b^2-c^2)(ab+ac)+2ab(b^2-c^2)-2ab(b^2-c^2).\cos C\geq 0$$

$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-(b^2-c^2)(ab+ac)+2ab(b^2-c^2).2\sin^2\frac{C}{2}\geq 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)-(b^2-c^2)(ab+ac)+(b^2-c^2)\left(c^2-(a-b)^2\right)\geq 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)+(b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac)\geq 0$$

$$\Leftrightarrow\left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$
$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac)\geq 0$$

$$\Leftrightarrow(c^2+a^2-b^2)(2b^2-ab-ac)+(b^2-c^2)(b^2+c^2-a^2-2ac)\geq 0$$

$$\Leftrightarrow(b^2-c^2)(b^2+c^2-2ac)-(b^2-c^2)(2b^2-ab-ac)$$
$$+a^2(2b^2-ab-ac)-a^2(b^2-c^2)\geq 0$$

$$\Leftrightarrow(b^2-c^2)\left((c^2-ca)-(b^2-ab)\right)+a^2\left((c^2-ca)+(b^2-ab)\right)\geq 0$$

$$\Leftrightarrow(c^2-ca)(b^2-c^2+a^2)+(b^2-ab)(a^2+c^2-b^2)\geq 0$$

$$\Leftrightarrow c(c-a)(a^2+b^2-c^2)+b(b-a)(c^2+a^2-b^2)\geq 0$$

$\to$ true $\because\Delta$ ABC being *acute* $\Rightarrow(a^2+b^2-c^2),(c^2+a^2-b^2)>0$ *and*

$$a=\min\{a,b,c\}\Rightarrow(c-a),(b-a)\ge 0\ \therefore\ \frac{b+c}{a}\ge\frac{R}{r}$$

$$\therefore\ \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}\ge\frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Rightarrow 2\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\cos\frac{B-C}{2}\ge 1\Rightarrow 2\cos^2\frac{B-C}{2}-2\cos\frac{B-C}{2}\sin\frac{A}{2}-1\overset{(\blacksquare)}{\ge}0$$

$$\text{We have } n_a+r_a\overset{CBS}{\le}\sqrt{2(n_a^2+r_a^2)}\overset{via\,(*)}{=}\sqrt{2(s^2-2r_ah_a+r_a^2)}\overset{?}{\le}2h_a$$

$$\Leftrightarrow s^2-2r_ah_a+s^2\tan^2\frac{A}{2}\overset{?}{\le}2h_a^2$$

$$\Leftrightarrow\frac{8s^2.16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}+\frac{4.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}s.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\overset{?}{\ge}s^2\sec^2\frac{A}{2}$$

$$\Leftrightarrow 8\sin^2\frac{B}{2}\sin^2\frac{C}{2}+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\overset{?}{\ge}1$$

$$\Leftrightarrow 2\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)^2+2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\overset{?}{\ge}1$$

$$\left(\because 2\sin\frac{B}{2}\sin\frac{C}{2}=\cos\frac{B-C}{2}-\cos\frac{B+C}{2}=\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)$$

$$\Leftrightarrow 2\left(\cos^2\frac{B-C}{2}+\sin^2\frac{A}{2}-2\sin\frac{A}{2}.\cos\frac{B-C}{2}\right)+2\sin\frac{A}{2}.\cos\frac{B-C}{2}$$

$$-2\sin^2\frac{A}{2}-1\overset{?}{\ge}0\Leftrightarrow 2\cos^2\frac{B-C}{2}-2\cos\frac{B-C}{2}\sin\frac{A}{2}-1\overset{?}{\ge}0\to\text{true via }(\blacksquare)$$

$$\therefore n_a+r_a\le 2h_a\le h_a+g_a\Rightarrow g_a+h_a\ge n_a+r_a>n_a$$

$$\therefore\ \boxed{\text{in acute }\Delta\,ABC\text{ with }a=\min\{a,b,c\},\text{holds}:g_a+h_a>n_a}\ (QED)$$

1324. In any Δ ABC and ∀ n ∈ ℕ, the following relationship holds :

$$\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\ge\frac{3^{n+1}.r^n}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{WLOG assuming } a\ge b\ge c\Rightarrow w_a^x\le w_b^x\le w_c^x\text{ and}$$

$$\frac{w_a^2}{(w_b+w_c)^2}\le\frac{w_b^2}{(w_c+w_a)^2}\le\frac{w_c^2}{(w_a+w_b)^2}\text{ and}$$

$$\frac{1}{(w_b+w_c)^2}\le\frac{1}{(w_c+w_a)^2}\le\frac{1}{(w_a+w_b)^2}\text{ where } x>0\to(1)$$

$$w_aw_bw_c\ge h_ah_bh_c=\frac{2r^2s^2}{R}\overset{Gerretsen}{\ge}\frac{r^2(27Rr+5r(R-2r))}{R}\overset{Euler}{\ge}\frac{r^2(27Rr)}{R}$$

$$\therefore w_aw_bw_c\ge 27r^3\to(2)$$

$$\sum_{cyc}w_a\ge\sum_{cyc}h_a=2rs\sum_{cyc}\frac{1}{a}\overset{Bergstrom}{\ge}\frac{2rs.9}{2s}\ \therefore\sum_{cyc}w_a\ge 9r\to(3)$$

Case 1 $n=1$ *and then*, $\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}$

$$=\sum_{cyc}\frac{w_a(w_a^2+w_bw_c)}{(w_b+w_c)^2}\overset{A-G}{\geq}\sum_{cyc}\frac{2w_a^2.\sqrt{w_bw_c}}{(w_b+w_c)^2}=2\sum_{cyc}\frac{\left(\frac{w_a}{w_b+w_c}\right)^2}{\frac{1}{\sqrt{w_bw_c}}}\overset{Bergstrom}{\geq}$$

$$\frac{2\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^2}{\sum_{cyc}\frac{1}{\sqrt{w_bw_c}}}\overset{\substack{Nesbitt\\and\\CBS}}{\geq}\frac{\frac{2.9}{4}}{\sqrt{\sum_{cyc}\frac{1}{w_b}}.\sqrt{\sum_{cyc}\frac{1}{w_c}}}\overset{w_a\geq h_a\text{ and analogs}}{\geq}\frac{\frac{9}{2}}{\sum_{cyc}\frac{1}{h_a}}=\frac{9r}{2}$$

$$\therefore\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}.r^n}{2}$$

Case 2 $n=2$ *and then*, $\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}$

$$=\sum_{cyc}\frac{w_a^2(w_a^2+w_bw_c)}{(w_b+w_c)^2}\overset{A-G}{\geq}\sum_{cyc}\frac{2w_a^3.\sqrt{w_bw_c}}{(w_b+w_c)^2}\overset{G-H}{\geq}\sum_{cyc}\frac{4w_a^3w_bw_c}{(w_b+w_c)^3}=4\sum_{cyc}\frac{\left(\frac{w_a}{w_b+w_c}\right)^3}{\frac{1}{w_bw_c}}$$

$$\overset{Holder}{\geq}\frac{4\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^3}{3\sum_{cyc}\frac{1}{w_bw_c}}\overset{Nesbitt}{\geq}\frac{\frac{4.27}{8}.w_aw_bw_c}{3\sum_{cyc}w_a}\geq\frac{9.\frac{16Rr^2s^2}{s^2+2Rr+r^2}}{2.\sqrt{3}s}\overset{?}{\geq}\frac{27r^2}{2}$$

$$\Leftrightarrow 256R^2s^2\underbrace{\overset{?}{\geq}}_{(*)}27(s^2+2Rr+r^2)^2\text{ and LHS of }(*)\overset{Mitrinovic}{\geq}\frac{256.4s^4}{27}$$

$$\overset{?}{\geq}27(s^2+2Rr+r^2)^2\Leftrightarrow 32s^2\overset{?}{\geq}27(s^2+2Rr+r^2)\Leftrightarrow 5s^2\overset{?}{\geq}27(2Rr+r^2)$$

$$\text{and }5s^2\overset{Gerretsen}{\geq}\frac{5}{2}(27Rr+5r(R-2r))\overset{Euler}{\geq}\frac{5}{2}(27Rr)\overset{?}{\geq}27(2Rr+r^2)\Leftrightarrow R\overset{?}{\geq}2r$$

$$\rightarrow\text{ true via Euler }\therefore\sum_{cyc}\frac{w_a^2(w_a^2+w_bw_c)}{(w_b+w_c)^2}\geq\frac{27r^2}{2}$$

$$\therefore\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}.r^n}{2}$$

Case 3 $n=3$ *and then*, $\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}$

$$=\sum_{cyc}\frac{w_a^3(w_a^2+w_bw_c)}{(w_b+w_c)^2}\overset{A-G}{\geq}\frac{2w_a^4.\sqrt{w_bw_c}}{(w_b+w_c)^2}\overset{G-H}{\geq}\sum_{cyc}\frac{4w_a^4w_bw_c}{(w_b+w_c)^3}$$

$$=4w_aw_bw_c\sum_{cyc}\left(\frac{w_a}{w_b+w_c}\right)^3\overset{Holder}{\geq}\frac{4}{9}.w_aw_bw_c.\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^3\overset{\substack{Nesbitt\\and\\(2)}}{\geq}\frac{4}{9}.27r^3.\frac{.27}{8}$$

$$=\frac{81r^3}{2}\therefore\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}.r^n}{2}$$

$$\boxed{\text{Case 4}}\ n \in \mathbb{N} - \{1,2,3\} \text{ and then, } \frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}$$

$$+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}=\sum_{cyc}\left(w_a^n.\left(\frac{w_a}{w_b+w_c}\right)^2\right)+w_aw_bw_c.\sum_{cyc}\frac{w_a^{n-1}}{(w_b+w_c)^2}$$

$$\overset{\substack{\text{Chebyshev, via (1)}\\ \text{and}\\ (2)}}{\geq}\frac{1}{3}\left(\sum_{cyc}w_a^n\right)\left(\sum_{cyc}\left(\frac{w_a}{w_b+w_c}\right)^2\right)$$

$$+9r^3.\left(\sum_{cyc}w_a^{n-1}\right)\left(\sum_{cyc}\frac{1^3}{(w_b+w_c)^2}\right)\overset{\substack{\text{Repeated Chebyshev}\\ \text{and}\\ \text{Radon}}}{\geq}\frac{1}{3.3.3^{n-1}}\left(\sum_{cyc}w_a\right)^n\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^2$$

$$+\frac{9r^3}{3^{n-2}}\left(\sum_{cyc}w_a\right)^{n-1}.\frac{27}{4\left(\sum_{cyc}w_a\right)^2}\overset{\text{Nesbitt}}{\geq}\frac{1}{3^{n+1}}.\frac{9}{4}.\left(\sum_{cyc}w_a\right)^n$$

$$+\frac{3^5r^3}{4.3^{n-2}}.\left(\sum_{cyc}w_a\right)^{n-3}\overset{\text{via (3)}}{\geq}\frac{1}{3^{n+1}}.\frac{9}{4}.3^{2n}.r^n+\frac{3^5r^3}{4.3^{n-2}}.3^{2n-6}.r^{n-3}$$

$$=\frac{3^{n+1}.r^n}{4}+\frac{3^{n+1}.r^n}{4}\ \therefore\ \sum_{cyc}\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}\geq\frac{3^{n+1}.r^n}{2}$$

$\therefore$ combining *all* cases, in *any* Δ ABC *and* $\forall\, n \in \mathbb{N}$,

$$\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}.r^n}{2},$$

" = " iff Δ ABC is equi*lateral* (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\frac{w_a{}^n(w_a{}^2+w_bw_c)}{(w_b+w_c)^2}=\frac{w_a{}^n}{(w_b+w_c)^2}(w_a-w_b)(w_a-w_c)+\frac{w_a{}^{n+1}}{w_b+w_c}\ \text{(and analogs)}$$

WLOG, we may assume that $w_a \geq w_b \geq w_c$.

$$\text{We have } \frac{w_a{}^n}{(w_b+w_c)^2}\geq\frac{w_b{}^n}{(w_c+w_a)^2}\geq\frac{w_c{}^n}{(w_a+w_b)^2},$$

then by the Generalized Schur inequality, we have

$$\sum_{cyc}\frac{w_a{}^n}{(w_b+w_c)^2}(w_a-w_b)(w_a-w_c)\geq 0.$$

Therefore

$$\sum_{cyc}\frac{w_a{}^n(w_a{}^2+w_bw_c)}{(w_b+w_c)^2}\geq\sum_{cyc}\frac{w_a{}^{n+1}}{w_b+w_c}\overset{\text{Hölder}}{\geq}\frac{(w_a+w_b+w_c)^{n+1}}{3^{n-1}.2(w_a+w_b+w_c)}\geq\frac{(h_a+h_b+h_c)^2}{6}$$

$$\overset{AM-HM}{\geq} \frac{1}{6}\left(\frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^2 = \frac{(9r)^2}{6} = \frac{27r^2}{2}.$$

as desired. Equality holds if and only if ΔABC is equilateral.

1325. In any Δ ABC and ∀ m, n ∈ ℕ such that : n ≥ m − 2,

the following relationship holds :

$$\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m} \geq \frac{3^{n-m+3}.\,r^{n-m+2}}{2^{m-1}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove that $\forall\, x, y, z > 0, \sum_{cyc}\frac{x^3+xyz}{(y+z)^3} \overset{(\blacklozenge)}{\geq} \frac{3}{4}$

$$\sum_{cyc}\frac{x^3+xyz}{(y+z)^3} = \sum_{cyc}\frac{x^3}{(y+z)^3}+xyz\sum_{cyc}\frac{1^4}{(y+z)^3} \overset{\substack{\text{Chebyshev}\\ \text{and}\\ \text{Radon}}}{\geq}$$

$$\frac{1}{3}\left(\sum_{cyc}\frac{x}{y+z}\right)\left(\sum_{cyc}\frac{x^2}{(y+z)^2}\right)+\frac{81xyz}{8\left(\sum_{cyc}x\right)^3} \overset{\text{Nesbitt}}{\geq} \frac{1}{3}\cdot\frac{3}{2}\cdot\sum_{cyc}\frac{x^4}{x^2y^2+x^2z^2+2x^2yz}$$

$$+\frac{81xyz}{8\left(\sum_{cyc}x\right)^3} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}x^2\right)^2}{4\left(\sum_{cyc}x^2y^2+xyz\sum_{cyc}x\right)}+\frac{81xyz}{8\left(\sum_{cyc}x\right)^3} \overset{?}{\geq} \frac{3}{4}$$

$$\Leftrightarrow \boxed{\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}x^2y^2+xyz\sum_{cyc}x}+\frac{81xyz}{2\left(\sum_{cyc}x\right)^3} \overset{?}{\underset{(\blacksquare)}{\geq}} 3}$$

Assigning $y+z=X, z+x=Y, x+y=Z \Rightarrow X+Y-Z=2z>0, Y+Z-X=2x > 0$ and $Z+X-Y=2y>0 \Rightarrow X+Y>Z, Y+Z>X, Z+X>Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc}x = \sum_{cyc}X = 2s \Rightarrow \sum_{cyc}x = s \to (1)$

$\Rightarrow x = s-X, y = s-Y, z = s-Z \Rightarrow xyz = r^2s \to (2)$ and also, such substitutions $\Rightarrow \sum_{cyc}xy = \sum_{cyc}(s-X)(s-Y) \Rightarrow \sum_{cyc}xy = 4Rr+r^2 \to (3)$ and

$$\sum_{cyc}x^2 = \left(\sum_{cyc}x\right)^2 - 2\sum_{cyc}xy \overset{\text{via (1) and (3)}}{=} s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \to (4) \text{ and also}, \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s.s \therefore \sum_{cyc} x^2y^2 = r^2((4R+r)^2 - 2s^2) \to (5)$$

$$\therefore \text{via } (1),(2),(4),(5), (\blacksquare) \Leftrightarrow \frac{(s^2 - 8Rr - 2r^2)^2}{r^2((4R+r)^2 - 2s^2) + r^2s.s} + \frac{81r^2s}{2s^3} \geq 3$$

$$\Leftrightarrow 2s^6 - (32Rr + 2r^2)s^4 + r^2s^2(32R^2 + 16Rr - 79r^2)$$

$$+r^4(1296R^2 + 648Rr + 81r^2) \overset{(\blacksquare\blacksquare)}{\geq} 0 \text{ and } \because 2(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } (\blacksquare\blacksquare), \text{it suffices to prove : LHS of } (\blacksquare\blacksquare)$$

$$\geq 2(s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (64Rr - 32r^2)s^4 - r^2s^2(1504R^2 - 976Rr + 229r^2)$$

$$+r^3(8192R^3 - 6384R^2r + 3048Rr^2 - 169r^3) \overset{(\blacksquare\blacksquare\blacksquare)}{\geq} 0 \text{ and}$$

$$\because (64Rr - 32r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0, \therefore \text{in order to prove } (\blacksquare\blacksquare\blacksquare),$$

$$\text{it suffices to prove : LHS of } (\blacksquare\blacksquare\blacksquare) \geq (64Rr - 32r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (544R^2 - 688Rr + 91r^2)s^2 \geq r(8192R^3 - 12048R^2r + 3672Rr^2 - 631r^3)$$

$$\Leftrightarrow 64t^3 - 210t^2 + 153t + 22 \geq 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)((t-2)(64t+46) + 81)$$

$$\geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\blacksquare\blacksquare\blacksquare) \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{x^3 + xyz}{(y+z)^3} \geq \frac{3}{4} \ \forall\, x,y,z > 0$$

$$\boxed{\text{Case 1}}\ n - m = -2 \text{ and then} : \frac{r_a^n(r_a^2 + r_br_c)}{(r_b + r_c)^m} + \frac{r_b^n(r_b^2 + r_cr_a)}{(r_c + r_a)^m} + \frac{r_c^n(r_c^2 + r_ar_b)}{(r_a + r_b)^m}$$

$$\geq \frac{3^{n-m+3}.\,r^{n-m+2}}{2^{m-1}} \Leftrightarrow \boxed{\sum_{cyc} \frac{r_a^{m-2}(r_a^2 + r_br_c)}{(r_b + r_c)^m} \overset{(*)}{\geq} \frac{3}{2^{m-1}}}$$

$$\text{Now}, n - m = -2 \Rightarrow m - 2 = n \geq 1 \Rightarrow m - 3 \geq 0 \therefore \sum_{cyc} \frac{r_a^{m-2}(r_a^2 + r_br_c)}{(r_b + r_c)^m}$$

$$= \sum_{cyc}\left(\left(\frac{r_a}{r_b + r_c}\right)^{m-3} \cdot \left(\frac{r_a^3 + r_ar_br_c}{(r_b + r_c)^3}\right)\right) \overset{\text{Chebyshev}}{\geq} \left(\sum_{cyc}\left(\frac{r_a}{r_b + r_c}\right)^{m-3}\right)\left(\sum_{cyc} \frac{r_a^3 + r_ar_br_c}{(r_b + r_c)^3}\right)$$

$$\left(\begin{array}{c} \because \text{WLOG assuming } a \geq b \geq c \Rightarrow \left(\frac{r_a}{r_b + r_c}\right)^{m-3} \geq \left(\frac{r_b}{r_c + r_a}\right)^{m-3}\left(\frac{r_c}{r_a + r_b}\right)^{m-3} \text{ and} \\ r_a^3 + r_ar_br_c \geq r_b^3 + r_ar_br_c \geq r_c^3 + r_ar_br_c \text{ alongwith } \frac{1}{(r_b + r_c)^3} \geq \frac{1}{(r_c + r_a)^3} \geq \frac{1}{(r_a + r_b)^3} \\ \Rightarrow \frac{r_a^3 + r_ar_br_c}{(r_b + r_c)^3} \geq \frac{r_b^3 + r_ar_br_c}{(r_c + r_a)^3} \geq \frac{r_c^3 + r_ar_br_c}{(r_a + r_b)^3} \end{array}\right)$$

$$\overset{\substack{\text{Repeated Chebyshev}\\ \text{and}\\ \text{via }(\blacklozenge)}}{\geq}\frac{1}{3}\cdot\frac{1}{3^{m-4}}\cdot\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^{m-3}\cdot\frac{3}{4}\overset{\text{Nesbitt}}{\geq}\frac{1}{4.3^{m-4}}\cdot\frac{3^{m-3}}{2^{m-3}}=\frac{3}{2^{m-1}}$$

$$\Rightarrow (*)\text{ is true }\therefore \frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

$\boxed{\textit{Case 2}}$ $n-m=-1$ *and then* : $\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}\Leftrightarrow\boxed{\sum_{cyc}\frac{r_a^{m-1}(r_a^2+r_br_c)}{(r_b+r_c)^m}\overset{(**)}{\geq}\frac{9r}{2^{m-1}}}$$

$$\text{Now, }\sum_{cyc}\frac{r_a^{m-1}(r_a^2+r_br_c)}{(r_b+r_c)^m}\overset{A-G}{\geq}2\sum_{cyc}\frac{r_a^m.\sqrt{r_br_c}}{(r_b+r_c)^m}=2\sum_{cyc}\frac{\left(\frac{r_a}{r_b+r_c}\right)^m}{\frac{1}{\sqrt{r_br_c}}}\overset{\text{Holder}}{\geq}$$

$$2.\frac{\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^m}{3^{m-2}.\sum_{cyc}\frac{1}{\sqrt{r_br_c}}}\overset{\substack{\text{Nesbitt}\\ \text{and}\\ \text{CBS}}}{\geq}\frac{2}{3^{m-2}}\cdot\frac{3^m}{2^m}\cdot\frac{1}{\sqrt{\sum_{cyc}\frac{1}{r_a}}.\sqrt{\sum_{cyc}\frac{1}{r_a}}}=\frac{9r}{2^{m-1}}\Rightarrow(**)\text{ is true}$$

$$\therefore\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

$\boxed{\textit{Case 3}}$ $n-m=0$ *and then* : $\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}\Leftrightarrow\boxed{\sum_{cyc}\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^n}\overset{(**)}{\geq}\frac{27r^2}{2^{n-1}}}$$

$$\text{Now, }\sum_{cyc}\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^n}\overset{A-G}{\geq}2\sum_{cyc}\frac{r_a^{n+1}.\sqrt{r_br_c}}{(r_b+r_c)^n}\overset{G-H}{\geq}4r_ar_br_c\sum_{cyc}\left(\left(\frac{r_a}{r_b+r_c}\right)^n\cdot\frac{1}{r_b+r_c}\right)\overset{\text{Chebyshev}}{\geq}$$

$$\frac{4rs^2}{3}\left(\sum_{cyc}\left(\frac{r_a}{r_b+r_c}\right)^n\right)\left(\sum_{cyc}\frac{1}{r_b+r_c}\right)\overset{\substack{\text{Repeated Chebyshev}\\ \text{and}\\ \text{Bergstrom}}}{\geq}\frac{4rs^2}{3.3^{n-1}}\cdot\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^n\cdot\frac{9}{2(4R+r)}$$

$$\overset{\text{Nesbitt}}{\geq}\frac{2rs^2}{3^n}\cdot\frac{3^n}{2^n}\cdot\frac{9}{4R+r}\overset{?}{\geq}\frac{27r^2}{2^{n-1}}\Leftrightarrow s^2\overset{?}{\geq}3r(4R+r)\to\text{true}$$

$$\because s^2\overset{\text{Gerretsen}}{\geq}3r(4R+r)+4r(R-2r)\overset{\text{Euler}}{\geq}3r(4R+r)$$

$$\Rightarrow(**)\text{ is true }\therefore\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

$\boxed{\textit{Case 4}}$ $n-m\geq 1$ *and then* : $\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$

$$\overset{A-G}{\geq}2\sum_{cyc}\frac{r_a^{n+1}.\sqrt{r_br_c}}{(r_b+r_c)^m}\overset{G-H}{\geq}4r_ar_br_c.\sum_{cyc}\frac{r_a^n}{(r_b+r_c)^{m+1}}$$

$$= 4rs^2.\sum_{cyc}\left(\left(\frac{r_a}{r_b+r_c}\right)^{m+1}.r_a^{n-m-1}\right)\overset{\text{Chebyshev}}{\geq}\frac{4rs^2}{3}\left(\sum_{cyc}\left(\frac{r_a}{r_b+r_c}\right)^{m+1}\right)\left(\sum_{cyc}r_a^{n-m-1}\right)$$

$$\overset{\text{Repeated Chebyshev}}{\geq}\frac{4rs^2}{3.3^m}\cdot\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^{m+1}\cdot\frac{1}{3^{n-m-2}}\cdot\left(\sum_{cyc}r_a\right)^{n-m-1}\overset{\substack{\text{Nesbitt}\\ \text{and}\\ \text{Euler}}}{\geq}$$

$$\frac{4rs^2}{3^{m+1}}\cdot\frac{3^{m+1}}{2^{m+1}}\cdot\frac{1}{3^{n-m-2}}\cdot(9r)^{n-m-1}\overset{\text{Mitrinovic}}{\geq}\frac{3^3.r^3.r^{n-m-1}}{2^{m-1}}\cdot\frac{1}{3^{n-m-2}}.3^{2n-2m-2}$$

$$=\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}\therefore\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$ $\therefore$ combining all cases, in any Δ ABC and $\forall$ m, n $\in \mathbb{N}$

such that : $n\geq m-2,\ \frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$," = " iff Δ ABC is equilateral (QED)

1326.

In any Δ ABC, the following relationships hold :

$$\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{b}{a}}\right)\geq\sum_{cyc}\sqrt{\frac{a}{b}};\ \frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right)\geq\sum_{cyc}\sqrt{\frac{w_a}{w_b}};\ \frac{R}{2r}\sum_{cyc}(g_a+h_a)>\sum_{cyc}n_a$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Tapas Das-India

$$n_a^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}$$

$$=s^2-\frac{4s(s-b)(s-c)}{a}=s^2-\frac{4s\cdot sr^2}{a(s-a)}=s^2-2r_ah_a\quad (1)$$

also $2r_a(n_a+h_a)\leq r_a^2+n_a^2+2r_ah_a=r_a^2+s^2$ (using (1))

(AM-GM)

$$=s^2\left(\tan^2\frac{A}{2}+1\right)=\frac{s^2}{\cos^2\frac{A}{2}}=\frac{s\cdot bc}{s-a}=\frac{r_a}{r}\cdot 2Rh_a$$

$$n_a+h_a\leq R\cdot\frac{h_a}{r}\text{ or }\frac{n_a}{h_a}\leq\frac{R}{r}-1$$

$$\therefore n_a\leq\left(\frac{R}{r}-1\right)h_a\quad\text{(analog)}$$

$$\therefore\sum n_a\leq\left(\frac{R}{r}-1\right)(h_a+h_b+h_c)$$

$$\frac{R}{2r}\cdot\sum(9a+h_a)\geq\frac{R}{2r}\sum(h_a+h_a)\quad(\because r_a\geq h_a)=\frac{R}{r}\sum h_a$$

We need to show

$$\frac{R}{r}\sum h_a\geq\left(\frac{R}{r}-1\right)(h_a+h_b+h_c)\text{ or }\frac{R}{r}\left(\sum h_a\right)-\left(\frac{R}{r}-1\right)\sum h_a\geq 0$$

$$\text{or }\sum h_a\geq 0\ \text{ (True)}$$

$$1)\ \sqrt{\frac{a}{b}}\overset{CBS}{\leq}\sqrt{\left(\sum a\right)\left(\sum\frac{1}{a}\right)}\leq\sqrt{2s\cdot\frac{9R}{4F}}=\sqrt{2s\cdot\frac{9R}{4rs}}=\sqrt{\frac{9R}{2r}}=\sqrt{\frac{9R^2}{2r\cdot R}}\overset{Euler}{\leq}\sqrt{\frac{9R^2}{9r^2}}=\frac{3R}{2r}$$

Now

$$\frac{R}{2r}\sum\sqrt{\frac{b}{a}}\geq\frac{3R}{2r}$$

(AM-GM)

$$\text{Note }\sqrt{\frac{b}{a}}\geq 3\text{ (AM-GM)}$$

$$\frac{R}{2r}\sum\sqrt{\frac{a}{b}}\geq\sum\sqrt{\frac{a}{b}}$$

$$\text{Note: }\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=\frac{s^2+r^2+4Rr}{4RF}\overset{Gerretsen}{\leq}\frac{4R^2+8Rr+4r^2}{4RF}=\frac{(R+r)^2}{RF}\overset{Euler}{\leq}\frac{9R^2}{4RF}=\frac{9R}{4F}$$

$$\sum\sqrt{\frac{w_a}{w_b}}\leq\sqrt{3\left(\sum\frac{w_a}{w_b}\right)}\leq\sqrt{\frac{3R}{2r}\cdot 3}=\sqrt{\frac{9R}{2r}}=\sqrt{\frac{9R^2}{2R\cdot r}}\overset{Euler}{\leq}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

Note:

$$\sum\frac{w_a}{w_b}\leq\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{w_b^2}\right)}\leq\sqrt{\sum(s(s-a))\sum\frac{1}{h_a^2}}\leq\sqrt{s^2\cdot\frac{\sum a^2}{4r^2s^2}}$$

$$\overset{Leibnitz}{\leq}\sqrt{\frac{s^2\cdot 9R^2}{9r^2s^2}}=\frac{3R}{2r}$$

We need to show

$$\frac{R^2}{4r^2}\sum\left(\sqrt{\frac{m_a}{m_b}}\right)\geq\frac{3R}{2r}\text{ or }\frac{R^2}{4r^2}\times 3\geq\frac{3R}{2r}$$

$$\left[\text{Note: }\sqrt{\frac{m_a}{m_b}}\geq 3\right]\text{ (AM-GM) or }\frac{R^2}{4r^2}\geq\frac{R}{2r}$$

$R \geq 2r$ (True) Euler

$$\therefore \frac{R^2}{4r^2}\left(\sum\sqrt{\frac{m_a}{m_b}}\right) \geq \frac{3R}{2r} \geq \sum\sqrt{\frac{w_a}{w_b}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{a}{b}} \overset{A-G}{\leq} \sum_{cyc}\frac{\frac{a}{b}+1}{2} = \sum_{cyc}\frac{a+b+c}{2b} - \frac{1}{2}\sum_{cyc}\frac{c}{b} \overset{A-G}{\leq} \frac{2s}{2}.\frac{s^2+4Rr+r^2}{4Rrs} - \frac{3}{2}$$

$$\overset{?}{\leq} \frac{3R}{2r} \Leftrightarrow \frac{3R+3r}{2r} \overset{?}{\geq} \frac{s^2+4Rr+r^2}{4Rr} \Leftrightarrow s^2+4Rr+r^2 \overset{?}{\leq} 6R^2+6Rr$$

$$\Leftrightarrow (s^2-4R^2-4Rr-3r^2) - 2(R+r)(R-2r) \overset{?}{\leq} 0 \rightarrow \text{true via Gerretsen and}$$

$$\text{Euler} \therefore \sum_{cyc}\sqrt{\frac{a}{b}} \leq \frac{3R}{2r} \overset{A-G}{\leq} \frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{b}{a}}\right) \therefore \boxed{\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{b}{a}}\right) \geq \sum_{cyc}\sqrt{\frac{a}{b}}}$$

$$\sum_{cyc}\sqrt{\frac{w_a}{w_b}} \overset{A-G}{\leq} \sum_{cyc}\frac{\frac{w_a}{w_b}+1}{2} = \sum_{cyc}\frac{w_a+w_b+w_c}{2w_b} - \frac{1}{2}\sum_{cyc}\frac{w_c}{w_b} \overset{A-G}{\leq} \frac{\sum_{cyc}w_a}{2}.\sum_{cyc}\frac{1}{w_a} - \frac{3}{2}$$

$$\leq \frac{\sum_{cyc}m_a}{2}.\sum_{cyc}\frac{1}{h_a} - \frac{3}{2} \overset{Bager}{\leq} \frac{4R+r}{2r} - \frac{3}{2} \overset{?}{\leq} \frac{3R^2}{4r^2} \Leftrightarrow \frac{3R^2+6r^2}{4r^2} \overset{?}{\geq} \frac{4R+r}{2r}$$

$$\Leftrightarrow 3R^2-8Rr+4r^2 \overset{?}{\geq} 0 \Leftrightarrow (3R-2r)(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true via Euler}$$

$$\therefore \sum_{cyc}\sqrt{\frac{w_a}{w_b}} \leq \frac{3R^2}{4r^2} \overset{A-G}{\leq} \frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right) \therefore \boxed{\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right) \geq \sum_{cyc}\sqrt{\frac{w_a}{w_b}}}$$

$$\text{Now, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2 + a(s-b)(s-c)$$

$$\text{and } b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2 + a(s-b)(s-c) \text{ and } (m)+(n) \Rightarrow$$

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$

$$= 2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2)$$

$$= 2(n_a^2+g_a^2) + a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2$$

$$= 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow (b-c)^2 + 4s(s-a) + (b-c)^2$$

$$= 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 \overset{(1)}{=} (b-c)^2 + 2s(s-a)$$

$$\text{Now,} \sum_{cyc}\frac{n_a}{R} = \sum_{cyc}\frac{2n_ah_a}{bc} \leq \sum_{cyc}\frac{2n_ag_a}{bc} \overset{A-G}{\leq} \sum_{cyc}\frac{n_a^2+g_a^2}{bc} \overset{\text{via }(1)}{=} \frac{(b-c)^2+2s(s-a)}{bc}$$

$$= \frac{1}{4Rrs}\left(\sum_{cyc}\left(a(b^2+c^2-2bc)\right) + 2s\sum_{cyc}(a(s-a))\right)$$

$$= \frac{1}{4Rrs}\left(2s(s^2+4Rr+r^2) - 36Rrs + 2s\left(2s^2 - 2(s^2-4Rr-r^2)\right)\right)$$

$$= \frac{s^2-6Rr+3r^2}{2Rr} \Rightarrow 2r\sum_{cyc}n_a \leq s^2-6Rr+3r^2 \overset{?}{<} 2R.\sum_{cyc}h_a = s^2+4Rr+r^2$$

$$\Leftrightarrow 9Rr + r(R-2r) \overset{?}{>} 0 \therefore 2r\sum_{cyc} n_a < 2R.\sum_{cyc} h_a \le R\sum_{cyc}(g_a + h_a)$$

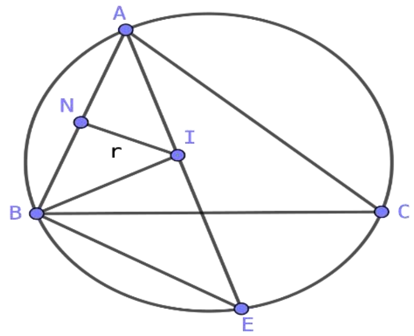
$$\therefore \boxed{\frac{R}{2r}\sum_{cyc}(g_a+h_a) > \sum_{cyc} n_a} \text{ (QED)}$$

1327. **Let $\Delta A'B'C'$ be the circumcevian triangle of incenter in acute ΔABC with r' - inradii. Prove that:**

$$r' \ge \frac{s}{3\sqrt{3}}, [A'B'C'] = \frac{1}{4}[I_aI_bI_c]$$

Proposed by Mehmet Şahin-Ankara-Turkiye

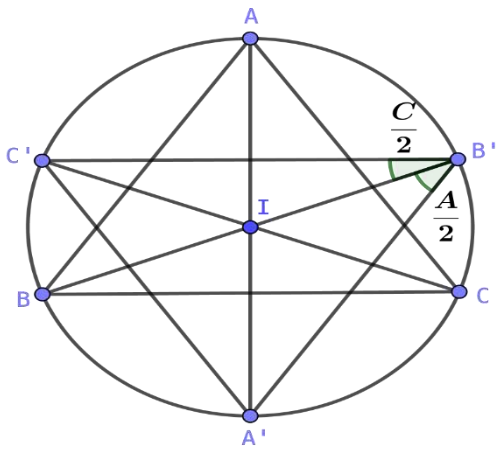
Solution by Tapas Das-India



$AI = r\csc\frac{A}{2}$. From ΔANI: $IN =$ perpendicular from I on $AB = r$ (in-radius)

Again $\angle BIE = \angle BAI + \angle ABI = \angle CAE + \angle IBC = \angle CBE + \angle IBC = \angle EBI$

$$\therefore IE = EB = 2R\sin\frac{A}{2}, \ \therefore IE = 2R\sin\frac{A}{2} \quad \text{(analog)}$$



From $\Delta IA'B'$

$$\angle AA'B' = \angle ABB' = \frac{B}{2}, \quad \angle BB'A' = \angle BAA' = \frac{A}{2}, \quad \angle A'IB' = \pi - \left(\frac{A+B}{2}\right)$$

From $\Delta A'IB'$

$$\frac{A'B'}{\sin \angle A'IB'} = \frac{A'I}{\sin\frac{A}{2}} \Rightarrow \frac{A'B'}{\sin\left(\pi - \frac{A+B}{2}\right)} = \frac{2R\sin\frac{A}{2}}{\sin\frac{A}{2}} = 2R$$

$$A'B' = 2R\sin\frac{A+B}{2} = 2R\cos\frac{C}{2}$$

$$[A'B'C'] = \frac{1}{2}B'C' \cdot A'B' \cdot \sin\angle A'B'C' = \frac{1}{2}2R\cos\frac{C}{2} \cdot 2R\cos\frac{A}{2} \cdot \sin\frac{A+C}{2}$$

$$[A'B'C'] = 2R^2\cos\frac{C}{2} \cdot \cos\frac{A}{2} \cdot \cos\frac{B}{2} \quad (1)$$

Now $[A'B'C'] = r's'$ (s' = semi – perimeter of $\Delta A'B'C'$)

or $2R^2\prod\cos\frac{A}{2} = \frac{r'}{2}2R\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)$

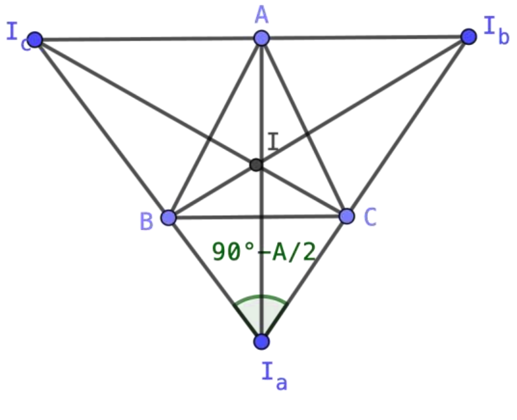
$$r' = \frac{2R\prod\cos\frac{A}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}, \qquad \therefore r' \geq \frac{2R \cdot \frac{s}{4R}}{\frac{3\sqrt{3}}{2}} = \frac{s}{3\sqrt{3}}$$

Note:

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq 3\cos\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) \overset{Jensen}{=} 3\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$$

$f(x) = \cos x \ \therefore f'(x) = -\sin x, f''(x) = -\cos x < 0, \therefore f$ is concave

2nd part:



Since ΔABC is orthic triangle of $\Delta I_aI_bI_c$ then

$$BC = a = I_aI_c \cdot \cos\left(90^\circ - \frac{A}{2}\right) = I_bI_c \sin\frac{A}{2}, \qquad a = 2R\sin A = I_bI_c \sin\frac{A}{2}$$

$$2R\,2 \cdot \sin\tfrac{A}{2}\cos\tfrac{A}{2} = I_bI_c \sin\tfrac{A}{2},\ \ I_bI_c = 4R\cos\tfrac{A}{2}\ \ \text{(analog)}$$

$$[I_aI_bI_c] = \frac{1}{2}\cdot 4R\cos\frac{B}{2}4R\cos\frac{C}{2}\sin\left(90^\circ - \frac{A}{2}\right) = 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$[A'B'C'] = 2R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\text{(From (1))} \therefore [A'B'C'] = \tfrac{1}{4}[I_aI_bI_c]$$

1328. In ΔABC the following relationship holds:

$$\sum\frac{w_a}{w_b} + \frac{R^3}{r^3} \geq 8 + \sum\frac{w_b}{w_a}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\sum\frac{w_b}{w_a} \leq \sum\frac{\sqrt{s(s-b)}}{w_a} \leq \sum\frac{\sqrt{s(s-b)}}{h_a}$$

$$\overset{CBS}{\leq} \sqrt{\left(\sum s(s-b)\right)\left(\sum\frac{1}{h_a^2}\right)} = \sqrt{s^2\cdot\frac{\sum a^2}{4r^2s^2}} \overset{Leibnitz's}{\leq} \sqrt{\frac{9R^2}{4r^2}} = \frac{3R}{2r}$$

$$\therefore 8 + \sum\frac{w_b}{w_a} \leq \frac{3R}{2r} + 8$$

Note: $\sum\frac{w_b}{w_a} \geq 3$ (AM-GM)

We need to show

$$3 + \tfrac{R^3}{r^3} \geq \tfrac{3R}{2r} + 8 \text{ or } \tfrac{R^3}{r^3} \geq \tfrac{3R}{2r} + 5$$

$$\text{or } x^3 \geq \tfrac{3}{2}x + 5 \quad \left(\tfrac{R}{r} = x \geq 2\right) \text{ or } 2x^3 - 3x - 10 \geq 0$$

$$2x^3 - 4x^2 + 4x^2 - 8x + 5x - 10 \geq 0$$

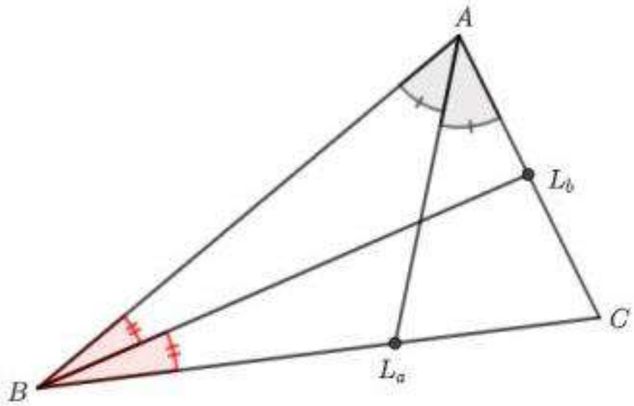
$$\text{or } 2x^2(x-2) + 4x(x-2) + 5(x-2) \geq 0$$

$$(x-2)(2x^2 + 4x + 5) \geq 0 \quad \text{(True)}$$

as $x \geq 2$.

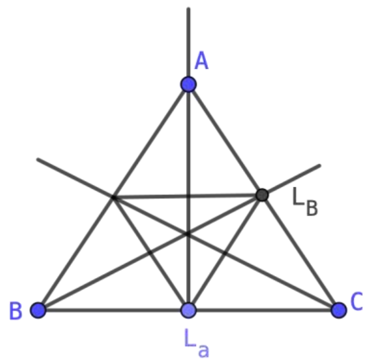
1329. Prove that:

$$L_aL_b = \frac{4F\sqrt{R(R+2r_c)}}{(a+c)(b+c)}$$



Proposed by Aissa Hiyab-Morocco

Solution by Tapas Das-India



$$\frac{BL_a}{CL_a} = \frac{c}{b}$$

$$\therefore\ CL_a = \frac{ab}{b+c}$$

$$CL_B = \frac{ab}{c+a}$$

$$\therefore\ L_aL_b^2 = \frac{a^2b^2}{(b+c)^2} + \frac{a^2b^2}{(c+a)^2} - 2\cos C\ CL_a \cdot CL_b$$

$$= \frac{a^2b^2}{(b+c)^2} + \frac{a^2b^2}{(c+a)^2} - 2\frac{a^2b^2}{(b+c)(c+a)} \cdot \frac{a^2+b^2-c^2}{2ab}$$

$$=\frac{a^2b^2(c+a)^2+a^2b^2(b+c)^2-ab(b+c)(c+a)(a^2+b^2-c^2)}{(b+c)^2(c+a)^2}$$

$$=\frac{a^2b^2(c^2+a^2+2ca+b^2+c^2+2bc)-ab(bc+ab+c^2+ac)(a^2+b^2-c^2)}{(b+c)^2(c+a)^2}$$

$$=\frac{a^2b^2(c^2+a^2+2ca+b^2+c^2+2bc)-a^2b^2(a^2+b^2-c^2)-ab(c^2+ac+bc)(a^2+b^2-c^2)}{(b+c)^2(c+a)^2}\quad (1)$$

Now, $a^2b^2(c^2+a^2+b^2+c^2+2ca+2bc)-a^2b^2(a^2+b^2-c^2)-$

$$-ab(c^2+ac+bc)(a^2+b^2-c^2)$$

$$=a^2b^2[c^2+2(bc+ca+c^2)]-(bc+ca+c^2)(a^2+b^2-c^2)ab$$

$$=a^2b^2c^2+(bc+ca+c^2)(2a^2b^2-(a^2+b^2-c^2)ab)$$

$$=a^2b^2c^2+2sc\cdot ab\,[2ab-(a^2+b^2-c^2)]$$

$$=a^2b^2c^2+2sc\cdot ab[c^2-(a-b)^2]$$

$$=a^2b^2c^2+2sc\cdot ab(c+a-b)(c-a+b)$$

$$=16F^2R^2+2s(4RF)(2s-2b)(2s-2a)=16F^2R^2+32sRF(s-b)(s-a)$$

$$=16F^2R^2+\frac{32F^2\cdot FR}{(s-c)}=16F^2R^2+32r_c\cdot F^2R$$

$$=16F^2R^2+32r_cF^2R=16F^2R(R+2r_c)$$

$\therefore$ From (1) we get,

$$L_aL_b^2=\frac{16F^2R(R+2r_c)}{(b+c)^2(c+a)^2},\qquad L_aL_b=\frac{4F\sqrt{R(R+2r_c)}}{(b+c)(c+a)}$$

1330. In ΔABC the following relationship holds:

$$\sum\frac{m_a}{m_bm_c(m_b+m_c)}\geq\frac{16r^3}{3R^5}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\sum\frac{m_a}{m_bm_c(m_b+m_c)}=\frac{1}{m_am_bm_c}\sum\frac{m_a^2}{m_b+m_c}$$

$$\geq\frac{1}{m_am_bm_c}\frac{(m_a+m_b+m_c)^2}{2(m_a+m_b+m_c)}=\frac{1}{2m_am_bm_c}(m_a+m_b+m_c)$$

Note:

$$m_a m_b m_c \le \frac{R^3 h_a h_b h_c}{8r^3} = \frac{R^3}{8r^3}\frac{8F^3}{abc} = \frac{R^3 F^3}{r^3 \cdot 4RF} = \frac{R^3 F^2}{4Rr^3}$$

Note: $m_a + m_b + m_c \ge 9r$

$$\therefore \sum \frac{m_a}{m_b m_c (m_b + m_c)} \ge \frac{1}{2 m_a m_b m_c}(m_a + m_b + m_c) \ge \frac{1}{2}\frac{4Rr^3}{R^3 F^2} \cdot 9r$$

$$\overset{Euler}{\ge} \frac{1}{2}\frac{4(2r)r^3 \cdot 9r}{R^3 r^2 s^2} \ge \frac{1}{2} \cdot \frac{4(2r)r^3 \cdot 9r}{R^3 r^2 \cdot \frac{27}{4} R^2}$$

$$\left(s^2 \le \frac{27}{4} R^2\right)$$

$$= \frac{1}{2} \cdot 8r \cdot 9r^4 \times \frac{4}{27 R^5 r^2} = \frac{16 r^3}{3 R^5}$$

Note:

$$m_a + m_b + m_c \ge \sqrt{s(s-a)} + \sqrt{s(s-b)} + \sqrt{s(s-c)}$$

$$\overset{AM-GM}{\ge} 3\left(\sqrt{s^3 (s-a)(s-b)(s-c)}\right)^{\frac{1}{3}} = 3\left(\sqrt{s^3 \cdot s r^2}\right)^{\frac{1}{3}} = 3(s^2 r)^{\frac{1}{3}}$$

$$(s^2 \ge 27r^2) \ge 3(27r^2 \cdot r)^{\frac{1}{3}} = 9r$$

1331. Prove that in any triangle ABC with usual notations, $x, y \in \mathbb{R}_+^*$, $xy = 1$, holds the following inequalities:

i) $(ax^2 + b + cy^2)(ay^2 + b + cx^2) \ge 4s^2$

ii) $(ax + b + cy)(ay + b + cx) \ge 4s^2$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Tapas Das – India

i) $(ax^2 + b + cy^2)(ay^2 + b + cx^2)$

$$\overset{Cauchy-Schwarz}{\ge} (axy + b + cxy)^2 = (a + b + c)^2 \ (\because xy = 1) = (2s)^2 = 4s^2$$

ii) $(ax + b + cy)(ay + b + cx) \ge \left(a\sqrt{xy} + b + c\sqrt{xy}\right)^2$ (Cauchy – Schwarz)

$$= (a + b + c)^2 = (2s)^2 = 4s^2$$

$$\because (xy = 1)$$

1332. In ΔABC the following relationships holds:

$$\left(\frac{\left(\sqrt{m_a}\right)^3+\left(\sqrt{w_a}\right)^3}{2}\right)^2 \ge \left(2\sqrt{s(s-a)}\right)^3$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\text{Note: } m_a \ge \frac{b+c}{2}\cos\frac{A}{2}$$

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2}$$

$$\therefore m_a w_a = \frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{2bc}{b+c}\cdot\cos\frac{A}{2} = bc\cos^2\frac{A}{2} = bc\frac{s(s-a)}{bc} = s(s-a)$$

$$\therefore \left[\frac{\left(\sqrt{m_a}\right)^3+\left(\sqrt{w_a}\right)^3}{2}\right]^2 \overset{AM-GM}{\ge} \left[\sqrt{\left(\sqrt{m_a}\right)^3\cdot\left(\sqrt{w_a}\right)^3}\right]^2 = \left(\sqrt{m_a\cdot w_a}\right)^3 \ge \left(\sqrt{s(s-a)}\right)^3$$

1333. In ΔABC the following relationship holds:

$$4r^2(5r-r) \le \sum r_a\cdot IA^2 \le 4r(R+r)^2$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\sum r_a I_a^2 = \sum r_a r^2\csc^2\frac{A}{2} = \sum r_a r^2\left(1+\cot^2\frac{A}{2}\right)$$

$$= \sum r_a r^2\left(1+\frac{s^2}{r_a^2}\right) = \sum (r_a r^2) + \left(\frac{r^2 s^2}{r_a}\right) = r^2\sum r_a + \sum\frac{r^2 s^2}{r\cdot s}(s-a)$$

$$= r^2\sum r_a + \sum rs(s-a) = r^2(4R+r) + (3s^2-2s^2)r = r^2(4R+r)+s^2 r$$

$$\overset{Gerretsen's}{\le} r^2(4R+r) + (4R^2+4Rr+3r^2)r$$

$$= r(4R^2+4Rr+3r^2+4Rr+r^2) = 4r(R+r)^2$$

$$\text{Again, } \sum r_a\cdot IA^2 = r^2(4R+r)+s^2 r \overset{Gerretsen's}{\ge} r^2(4R+r)+r(16Rr-5r^2)$$

$$= r[4Rr+r^2+16Rr-5r^2] = r[20Rr-4r^2] = 4r^2(5R-r)$$

1334. In ΔABC the following relationship holds:

$$\sum a^{2n} \ge 3\left(\frac{4F}{3}\sqrt{\sum \frac{a^2}{b^2}}\right)^n, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

Oppenheim, $\forall x, y, z > 0$ in ΔABC

$$a^2x + b^2y + c^2z \ge 4F\sqrt{xy + yz + zx}$$

Let $x = \frac{b^2}{a^2}, y = \frac{c^2}{b^2}, z = \frac{a^2}{c^2}$

$$\therefore a^2 + b^2 + c^2 = a^2 \cdot \frac{b^2}{a^2} + b^2 \cdot \frac{c^2}{b^2} + c^2 \cdot \frac{a^2}{c^2} \ge 4F\sqrt{\frac{b^2}{a^2} \cdot \frac{c^2}{b^2} + \frac{c^2}{b^2} \cdot \frac{a^2}{c^2} + \frac{a^2}{c^2} \cdot \frac{b^2}{a^2}}$$

$$= 4F\sqrt{\frac{c^2}{a^2} + \frac{a^2}{b^2} + \frac{b^2}{c2}} \quad (1)$$

$$\therefore a^{2n} + b^{2n} + c^{2n} = (a^2)^n + (b^2)^n + (c^2)^n \overset{CBS}{\ge} \frac{3(a^2 + b^2 + c^2)^n}{3^n}$$

$$\ge 3\left[\frac{4F}{3}\sqrt{\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}}\right]^n \text{(Using (1))}$$

1335. In acute ΔABC the following relationship holds:

$$\frac{2R^2}{r^2} + 1 \le \sum \frac{(1 + \sec A)^2}{\tan^2 A} \le \left(\frac{2R}{r} - 1\right)^2$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

1st part:

$$\sum \frac{(1 + \sec A)^2}{\tan^2 A} = \sum \frac{(1 + \sec A)^2}{(\sec^2 A - 1)} = \sum \frac{1 + \sec A}{\sec A - 1} = \sum \frac{1 + \cos A}{1 - \cos A} = \sum \cot\frac{A}{2}$$

$$\therefore \sum \cot^2\frac{A}{2} = \sum \frac{1}{\tan^2\frac{A}{2}} = \frac{\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\prod \tan^2\frac{A}{2}} = \frac{1 - \frac{2r^2 + 8Rr}{s^2}}{\frac{r^2}{s^2}}$$

$$= \frac{s^2 - 2r^2 - 8Rr}{r^2} \overset{Gerretsen's}{\le} \frac{4R^2 - 4Rr + r^2}{r^2} = \left(\frac{2R}{r} - 1\right)^2$$

2nd part:

$$\sum\frac{(1+\sec A)^2}{\tan^2 A}=\sum\cot^2\frac{A}{2}=\sum\frac{1}{\tan^2\frac{A}{2}}=\frac{\sum\tan^2\frac{A}{2}\cdot\tan^2\frac{B}{2}}{\prod\tan^2\frac{A}{2}}=\frac{s^2-2r^2-8Rr}{r^2}$$

$$\overset{\substack{(Walker's)\\ since\ triangle\ is\ acute}}{\geq}\frac{2R^2+8Rr+3r^2-2r^2-8Rr}{r^2}=\frac{2R^2}{r^2}+1$$

1336. In any acute triangle ABC holds :

$$m_a\sqrt{\cot A}+m_b\sqrt{\cot B}+m_c\sqrt{\cot C}>6r,$$

where m_a, m_b, m_c are the medians and r is the inradius of the triangle.

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we may assume that $a\leq b\leq c$.
We have $m_a\geq m_b\geq m_c$ and $\sqrt{\cot A}\geq\sqrt{\cot B}\geq\sqrt{\cot C}$,
then by Chebyshev's inequality, we get

$$m_a\sqrt{\cot A}+m_b\sqrt{\cot B}+m_c\sqrt{\cot C}\geq\frac{1}{3}(m_a+m_b+m_c)\left(\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\right).$$

Since $m_a\geq h_a$ (and analogs) and
$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$, then by using AM − HM inequality, we have

$$m_a+m_b+m_c\geq h_a+h_b+h_c\geq\frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=9r.$$

Now, let $x:=\cot A$, $y:=\cot B$, $z:=\cot C$.
We have $x\geq y\geq z>0$, $xy+yz+zx=1$ and by using
AM − GM inequality, we obtain

$$\left(\frac{\sqrt{x}+\left(\sqrt{y}+\sqrt{z}\right)}{2}\right)^4\geq\left(\sqrt{x}\left(\sqrt{y}+\sqrt{z}\right)\right)^2=xy+2x\sqrt{yz}+zx>xy+yz+zx=1,$$

then $\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}=\sqrt{x}+\sqrt{y}+\sqrt{z}>2$.

Therefore

$$m_a\sqrt{\cot A}+m_b\sqrt{\cot B}+m_c\sqrt{\cot C}>\frac{1}{3}.9r.2=6r.$$

1337. O – the circumcenter of ΔABC lies on the incircle of ΔABC. Prove that:

$$8\sqrt{2}+\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}>12$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

$$\cos\frac{B-C}{2}=\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\qquad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

Using Ravi's transformation

$$a=y+z, b=z+x, c=x+y$$

Now we need to show $\cos\frac{B-C}{2}\ge\sqrt{\frac{2r}{R}}$

Now the inequality is equivalent to with $s=x+y+z$

$$\cos\frac{B-C}{2}=\sqrt{\frac{(s-b)(s-c)}{ac\cdot ab}}+\sqrt{\frac{(s-a)(s-c)}{ac}\cdot\frac{(s-b)(s-a)}{ab}}$$

This is equivalent to: $(2x+y+z)^2\ge 8x(y+z)$

This is true using AM-GM

$$\therefore \cos\frac{B-C}{2}\ge\sqrt{\frac{2r}{R}}$$

$$\therefore \cos\frac{B-C}{2}\cdot\cos\frac{A-B}{2}\cdot\cos\frac{C-A}{2}\ge\frac{2r}{R}\sqrt{\frac{2r}{R}}$$

Now incircle passes through circumcentre

$$\therefore OI=r\Rightarrow\sqrt{R^2-2Rr}=r$$

$$\therefore\left(\frac{r}{R}\right)^2+2\left(\frac{r}{R}\right)-1=0$$

$$\therefore\frac{r}{R}=\frac{-2\pm\sqrt{4+4}}{2}\ \therefore\frac{r}{R}=\left(\sqrt{2}-1\right)\quad\left(\frac{r}{R}>0\right)$$

We need to show

$$8\sqrt{2}+\prod\cos\frac{A-B}{2}>12 \text{ or } 8\sqrt{2}+\frac{2r}{R}\sqrt{\frac{2r}{R}}>12$$

$$\text{or } 8\sqrt{2}+2\left(\sqrt{2}-1\right)\sqrt{2\left(\sqrt{2}-1\right)}>12 \text{ or } 4\sqrt{2}+\left(\sqrt{2}-1\right)\sqrt{2\left(\sqrt{2}-1\right)}>6$$

or $4+(\sqrt{2}-1)\sqrt{\sqrt{2}-1}>3\sqrt{2}$ or $(4-3\sqrt{2})>-(\sqrt{2}-1)\sqrt{\sqrt{2}-1}$

or $16+18-24\sqrt{2}>(\sqrt{2}-1)^2(\sqrt{2}-1)$ or $34-24\sqrt{2}>3\sqrt{2}-7+2\sqrt{2}$

or $41>29\sqrt{2}$ True.

$$\prod\cos\frac{A-B}{2}=\frac{(a+b)(b+c)(c+a)}{abc}\prod\sin\frac{A}{2}$$

$$=\frac{(a+b+c)(ab+bc+ca)-abc}{abc}\cdot\frac{r}{4R}=\frac{s^2+r^2+2Rr}{8R^2}$$

When incircle passes through circumcircle

$OI=r\Rightarrow\sqrt{R^2-2Rr}=r$ or $\left(\frac{r}{R}\right)^2+2\left(\frac{r}{R}\right)-1=0$

or $\frac{r}{R}=\frac{-2\pm\sqrt{4+4}}{2}\therefore\frac{r}{R}=\sqrt{2}-1$

$$\left(\frac{r}{R}>0\right)$$

We need to show:

$$8\sqrt{2}+\frac{s^2+r^2+2Rr}{8R^2}>12$$

Or $8\sqrt{2}+\frac{27r^2+r^2+2Rr}{8R^2}>12$ (Mitrinovic)

or $8\sqrt{2}+\frac{27(\sqrt{2}-1)^2\cdot R^2+(\sqrt{2}-1)^2R^2+2R^2(\sqrt{2}-1)}{8R^2}>12$

(using $\frac{R}{r}=\sqrt{2}-1$)

or $64\sqrt{2}+28(\sqrt{2}-1)^2+2(\sqrt{2}-1)>96$ or $64\sqrt{2}+28(3-2\sqrt{2})+2\sqrt{2}-2>96$

or $64\sqrt{2}+84-56\sqrt{2}+2\sqrt{2}-2>96$ or $10\sqrt{2}+82>96$ or $10\sqrt{2}>14$

or $(10\sqrt{2})^2>(14)^2$ or $250>196$ (True)

$\cos\frac{B-C}{2}=\frac{h_a}{w_a}$ (analog)

We show that

$\prod\cos\frac{B-C}{2}\geq 8\prod\sin\frac{A}{2}$ or $\frac{h_ah_bh_c}{w_aw_bw_c}\geq 8\cdot\frac{r}{4R}$ or $Rh_ah_bh_c\geq 2r(w_aw_bw_c)$

or $R\cdot\frac{(2F)^3}{abc}\geq 2r\sqrt{s(s-a)}\sqrt{s(s-b)}\sqrt{s(s-c)}$ or $8F^3R\geq 2r^2s^2\cdot 4Rrs$

or $8r^3s^3R\geq 8r^3s^3R$ (True)

$$\therefore \prod \cos\frac{B-C}{2} \geq 8\prod \sin\frac{A}{2} = 2\cdot\frac{r}{R}$$
$$= 2(\sqrt{2}-1)$$

Now, incircle passes through circumcircle

$$\therefore OI = r \Rightarrow \sqrt{R^2-2Rr} = r \Rightarrow \left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right) - 1 = 0$$
$$\therefore \frac{r}{R} = \frac{-2\pm\sqrt{4+4}}{2} \therefore \frac{r}{R} = \sqrt{2}-1$$
$$\left(\frac{r}{R} > 0\right)$$

We need to show

$$8\sqrt{2} + \prod \cos\frac{B-C}{2} > 12 \text{ or } 8\sqrt{2} + 2(\sqrt{2}-1) > 12$$
$$\text{or } 10\sqrt{2} > 14 \text{ or } 250 > 196 \quad \text{(True)}$$

1338. In any Δ ABC and $\forall$ n $\in \mathbb{N}$, the following relationship holds :

$$\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)} + \frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)} + \frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)} \geq \frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{m_a} \overset{\text{Bergstrom}}{\geq} \frac{9}{\sum_{cyc} m_a} \overset{\text{Leuenberger}}{\geq} \frac{9}{4R+r} \overset{\text{Euler}}{\geq} \frac{9}{\frac{9R}{2}} \Rightarrow \sum_{cyc}\frac{1}{m_a} \overset{(i)}{\geq} \frac{2}{R}$$

$\boxed{Case\ 1}$ $n = 1$ *and then* : $\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)} + \frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)} + \frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)}$

$$= \sum_{cyc}\frac{m_a}{m_b+m_c} + \sum_{cyc}\frac{m_bm_c}{m_am_b+m_am_c} \overset{\text{Nesbitt}}{\geq} \frac{3}{2}+\frac{3}{2} = 3 = \frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1}$$

$\boxed{Case\ 2}$ $n \in \mathbb{N}-\{1\}$ *and then* : $\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)} + \frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)} + \frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)}$

$$= \sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^{n-2}}{m_b+m_c} + \sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^{n}}{\frac{m_b+m_c}{m_bm_c}} \overset{\text{Holder}}{\geq} \frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^{n-2}}{3^{n-4}*2*\sum_{cyc} m_a} + \frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^{n}}{3^{n-2}*2*\sum_{cyc}\frac{1}{m_a}}$$

$$\overset{\substack{\text{via (i)}\\ and \\ \text{Leuenberger + Euler}}}{\geq} \frac{\left(\frac{2}{R}\right)^{n-2}}{3^{n-4}*2*\frac{9R}{2}} + \frac{\left(\frac{2}{R}\right)^{n-1}}{3^{n-2}*2} = \frac{2^{n-2}}{3^{n-2}*R^{n-1}} + \frac{2^{n-2}}{3^{n-2}*R^{n-1}}$$

$$=\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1} \text{ and combining both cases, in any } \Delta\, ABC \text{ and } \forall\, n\in\mathbb{N},$$

$$\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)}$$

$$\geq\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1}, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we may assume that $m_a\geq m_b\geq m_c$.

Since

$$\frac{1}{m_a{}^n(m_b+m_c)}\leq\frac{1}{m_b{}^n(m_c+m_a)}\leq\frac{1}{m_c{}^n(m_a+m_b)},$$

then by the Generalized Schur inequality, we have

$$\sum_{cyc}\frac{m_a{}^2+m_bm_c}{m_a{}^n(m_b+m_c)}=\sum_{cyc}\left(\frac{(m_a-m_b)(m_a-m_c)}{m_a{}^n(m_b+m_c)}+\frac{1}{m_a{}^{n-1}}\right)\geq\sum_{cyc}\frac{1}{m_a{}^{n-1}}$$

$$\overset{\text{Hölder}}{\geq}\frac{3^n}{(m_a+m_b+m_c)^{n-1}}\overset{\text{Gotman}}{\geq}\frac{3^n}{\left(\frac{9R}{2}\right)^{n-1}}=\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1},$$

as desired. Equality holds iff ΔABC is equilateral.

1339. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2+m_bm_c}{m_a^5(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^5(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^5(m_a+m_b)}\geq\frac{16}{27R^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2+m_bm_c}{m_a^5(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^5(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^5(m_a+m_b)}$$

$$=\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^3}{m_b+m_c}+\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^5}{\frac{m_b+m_c}{m_bm_c}}\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^3}{6\sum_{cyc}m_a}+\frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^5}{27*2\sum_{cyc}\frac{1}{m_a}}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{\left(\frac{9}{\sum_{cyc}m_a}\right)^3}{6\sum_{cyc}m_a}+\frac{\left(\frac{9}{\sum_{cyc}m_a}\right)^4}{54}\overset{\text{Leuenberger}}{\geq}\frac{729}{6(4R+r)^4}+\frac{729*9}{54(4R+r)^4}$$

$$\overset{\text{Euler}}{\geq}\frac{729*2^4}{6*729*9R^4}+\frac{729*9*2^4}{54*729*9R^4}=\frac{16}{27R^4}, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sum_{cyc}\frac{m_a^2+m_bm_c}{m_a^5(m_b+m_c)}\ge\sum_{cyc}\frac{2m_a\sqrt{m_bm_c}}{m_a^5(m_b+m_c)}$$

$$\ge 2.3\sqrt[3]{\frac{m_a^2m_b^2m_c^2}{m_a^5m_b^5m_c^5(m_a+m_b)(m_b+m_c)(m_c+m_a)}}$$

$$=\frac{6}{m_am_bm_c\sqrt[3]{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}$$

$$\ge\frac{6}{\left(\frac{m_a+m_b+m_c}{3}\right)^3\cdot\frac{(m_a+m_b)+(m_b+m_c)+(m_c+m_a)}{3}}=\frac{243}{(m_a+m_b+m_c)^4}$$

$$\overset{Leuenberger}{\ge}\frac{243}{(4R+r)^4}\overset{Euler}{\ge}\frac{243}{\left(\frac{9R}{2}\right)^4}=\frac{16}{27R^4},$$

as desired. Equality holds iff ΔABC is equilateral.

1340. In any Δ ABC, the following relationship holds :

$$\frac{w_a(w_b^2+w_c^2)}{w_a^2+w_bw_c}+\frac{w_b(w_c^2+w_a^2)}{w_b^2+w_cw_a}+\frac{w_c(w_a^2+w_b^2)}{w_c^2+w_aw_b}\ge 9r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

Firstly, we shall prove that $\forall\, x,y,z>0$, $\prod_{cyc}(y^2+z^2)\ge\prod_{cyc}(x^2+yz)$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^2y^4\overset{(i)}{\ge}xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3$$

$$\text{LHS of (i)}=\sum_{cyc}\frac{x^4y^2+x^4z^2}{2}+\sum_{cyc}\frac{x^4y^2+x^2y^4}{2}\overset{A-G}{\ge}\sum_{cyc}x^4yz+\sum_{cyc}x^3y^3$$

$$=xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3\Rightarrow\text{(i) is true}\therefore\forall\,x,y,z>0,\frac{\prod_{cyc}(y^2+z^2)}{\prod_{cyc}(x^2+yz)}\ge 1\to(1)$$

$$\text{Now, }\frac{w_a(w_b^2+w_c^2)}{w_a^2+w_bw_c}+\frac{w_b(w_c^2+w_a^2)}{w_b^2+w_cw_a}+\frac{w_c(w_a^2+w_b^2)}{w_c^2+w_aw_b}\overset{A-G}{\ge}$$

$$3\sqrt[3]{w_aw_bw_c\cdot\frac{\prod_{cyc}(w_b^2+w_c^2)}{\prod_{cyc}(w_a^2+w_bw_c)}}\overset{via\ (1)}{\ge}3.\sqrt[3]{w_aw_bw_c}\ge 3.\sqrt[3]{h_ah_bh_c}=3.\sqrt[3]{\frac{2r^2s^2}{R}}$$

$$\overset{Gerretsen}{\ge}3.\sqrt[3]{\frac{r^2.(27Rr+5r(R-2r))}{R}}\overset{Euler}{\ge}3.\sqrt[3]{\frac{r^2.27Rr}{R}}=9r\therefore\text{ in any }\Delta\text{ ABC},$$

$$\frac{w_a(w_b^2+w_c^2)}{w_a^2+w_bw_c}+\frac{w_b(w_c^2+w_a^2)}{w_b^2+w_cw_a}+\frac{w_c(w_a^2+w_b^2)}{w_c^2+w_aw_b}\geq 9r,$$
$$''=''\text{ iff }\Delta\, ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\sum_{cyc}\frac{w_a(w_b^2+w_c^2)}{w_a^2+w_bw_c}\overset{CBS}{\geq}\sum_{cyc}\frac{w_a(w_b^2+w_c^2)}{\sqrt{(w_a^2+w_b^2)(w_a^2+w_c^2)}}\overset{AM-GM}{\geq}3\sqrt[3]{\prod_{cyc}\frac{w_a(w_b^2+w_c^2)}{\sqrt{(w_a^2+w_b^2)(w_a^2+w_c^2)}}}$$
$$=3\sqrt[3]{w_aw_bw_c}\overset{w_a\geq h_a\text{ (and analogs)}}{\geq}3\sqrt[3]{h_ah_bh_c}\overset{GM-HM}{\geq}\frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=9r.$$

as desired. Equality holds iff ΔABC is equilateral.

1341. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{m_a}{m_b\left(5(m_a^2+m_b^2)+m_a(6m_b+11m_c)\right)}\geq\frac{4}{81R^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0,\sum_{cyc}\frac{x}{y\left(5(x^2+y^2)+x(6y+11z)\right)}$$
$$=\sum_{cyc}\frac{x^2}{5x^3y+5xy^3+6x^2y^2+11x.xyz}\overset{\text{Bergstrom}}{\geq}$$
$$\frac{\left(\sum_{cyc}x\right)^2}{5\sum_{cyc}x^3y+5\sum_{cyc}xy^3+6\sum_{cyc}x^2y^2+11xyz\sum_{cyc}x}\overset{?}{\geq}\frac{1}{\left(\sum_{cyc}x\right)^2}$$
$$\Leftrightarrow\left(\sum_{cyc}x\right)^4\overset{?}{\geq}5\sum_{cyc}\left(xy\left(\sum_{cyc}x^2-z^2\right)\right)+6\sum_{cyc}x^2y^2+11xyz\sum_{cyc}x$$
$$\Leftrightarrow\left(\sum_{cyc}x\right)^4\underset{(*)}{\overset{?}{\geq}}5\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+6\sum_{cyc}x^2y^2+6xyz\sum_{cyc}x$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=$ s, R, r (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\rightarrow(1)\Rightarrow x=s-X, y=s-Y,$

$$z = s - Z \text{ and such substitutions} \Rightarrow \sum_{cyc} xy = \sum_{cyc} (s - X)(s - Y)$$

$$\Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (2) \text{ and } \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (1) and (2)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \to (3) \text{ and also}, \sum_{cyc} x^2y^2$$

$$= \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via (1) and (2)}}{=} (4Rr + r^2)^2 - 2\left(\prod_{cyc}(s - X)\right).s$$

$$= (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (4)$$

$$\therefore \text{via } (1),(2),(3),(4),(*) \Leftrightarrow s^4 \ge 5(4Rr + r^2)(s^2 - 8Rr - 2r^2)$$

$$+6r^2\left((4R + r)^2 - 2s^2\right) + 6\left(\prod_{cyc}(s - X)\right)s$$

$$= 5(4Rr + r^2)(s^2 - 8Rr - 2r^2) + 6r^2\left((4R + r)^2 - 2s^2\right) + 6r^2s$$

$$\Leftrightarrow s^4 - (20Rr - r^2)s^2 + 4r^2(4R + r)^2 \overset{(**)}{\ge} 0 \text{ and } \because (s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0$$

$$\therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \ge (s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (4R - 3r)s^2 \overset{(***)}{\ge} r(64R^2 - 64Rr + 7r^2)$$

$$\text{Now, } (4R - 3r)s^2 \overset{\text{Rouche}}{\ge} (4R - 3r)\left(2R^2 + 10Rr - r^2 - 2(R - 2r) * \sqrt{R^2 - 2Rr}\right)$$

$$\overset{?}{\ge} r(64R^2 - 64Rr + 7r^2) \Leftrightarrow (4R - 3r)(2R^2 + 10Rr - r^2) - r(64R^2 - 64Rr + 7r^2)$$

$$\overset{?}{\ge} 2(4R - 3r)(R - 2r) * \sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow 2(R - 2r)(4R^2 - 7Rr + r^2) \overset{?}{\ge} 2(4R - 3r)(R - 2r) * \sqrt{R^2 - 2Rr} \text{ and}$$

$$\text{proving it will be done if we can prove : } 4R^2 - 7Rr + r^2 > (4R - 3r) * \sqrt{R^2 - 2Rr}$$

$$\left(\because R - 2r \overset{\text{Euler}}{\ge} 0\right) \Leftrightarrow (4R^2 - 7Rr + r^2)^2 > (R^2 - 2Rr)(4R - 3r)^2$$

$$\Leftrightarrow 4Rr^3 + r^4 > 0 \to \text{true} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \forall x, y, z > 0, \sum_{cyc} \frac{x}{y\left(5(x^2 + y^2) + x(6y + 11z)\right)} \ge \frac{1}{\left(\sum_{cyc} x\right)^2}$$

$$\Rightarrow \sum_{cyc} \frac{m_a}{m_b\left(5(m_a^2 + m_b^2) + m_a(6m_b + 11m_c)\right)} \ge \frac{1}{\left(\sum_{cyc} m_a\right)^2} \ge \frac{1}{3\sum_{cyc} m_a^2}$$

$$= \frac{4}{9\sum_{cyc} a^2} \overset{\text{Leibnitz}}{\ge} \frac{4}{81R^2} \therefore \text{in any } \Delta\, ABC,$$

$$\sum_{cyc} \frac{m_a}{m_b\left(5(m_a^2 + m_b^2) + m_a(6m_b + 11m_c)\right)}$$

$$\ge \frac{4}{81R^2}, \text{"="} \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x,y,z>0$. By CBS inequality, we have

$$\sum_{cyc}\frac{x}{y[5(x^2+y^2)+x(6y+11z)]}\ge\frac{(x+y+z)^2}{\sum_{cyc}xy[5(x^2+y^2)+x(6y+11z)]}$$

$$=\frac{(x+y+z)^2}{3\left(\sum_{cyc}xy\right)^2+5\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+3\sum_{cyc}x^2y^2}$$

$$\ge\frac{(x+y+z)^2}{2\left(\sum_{cyc}x^2\right)^2+6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+\left(\sum_{cyc}x^2\right)^2}=\frac{1}{3(x^2+y^2+z^2)}.$$

Then by using Leibniz's inequality, we get

$$\sum_{cyc}\frac{m_a}{m_b[5({m_a}^2+{m_b}^2)+m_a(6m_b+11m_c)]}\ge\frac{1}{3({m_a}^2+{m_b}^2+{m_c}^2)}$$

$$=\frac{4}{9(a^2+b^2+c^2)}\ge\frac{4}{81R^2},$$

as desired. Equality holds iff ΔABC is equilateral.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\sum_{cyc}\frac{m_a}{m_b[5({m_a}^2+{m_b}^2)+m_a(6m_b+11m_c)]}\overset{CBS}{\ge}\frac{\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right)^2}{\sum_{cyc}[5({m_a}^2+{m_b}^2)+m_a(6m_b+11m_c)]}$$

$$\overset{AM-GM}{\ge}\frac{3^2}{10\sum_{cyc}{m_a}^2+17\sum_{cyc}m_bm_c}\ge\frac{9}{27\sum_{cyc}{m_a}^2}=\frac{4}{9\sum_{cyc}a^2}\overset{Leibniz}{\ge}\frac{4}{81R^2},$$

as desired. Equality holds iff ΔABC is equilateral.

1342. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{r_a}{r_b\left(6(r_a^3+r_b^3)+10r_ar_b(r_a+r_b)+r_ar_c(19r_a+30r_b)\right)}\ge\left(\frac{2}{9R}\right)^3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0,\sum_{cyc}\frac{x}{y\left(6(x^3+y^3)+10xy(x+y)+xz(19x+30y)\right)}$$

$$=\sum_{cyc}\frac{x^2}{6x^4y+6xy^4+10x^3y^2+10x^2y^3+xyz(19x^2+30xy)}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{\left(\sum_{cyc}x\right)^2}{6\sum_{cyc}x^4y+6\sum_{cyc}xy^4+10\sum_{cyc}x^3y^2+10\sum_{cyc}x^2y^3+xyz\left(19\sum_{cyc}x^2+30\sum_{cyc}xy\right)}$$

$$\overset{?}{\geq}\frac{1}{\left(\sum_{cyc}x\right)^3}\Leftrightarrow\left(\sum_{cyc}x\right)^5\overset{?}{\geq}6\sum_{cyc}\left(xy\left(\sum_{cyc}x^3-z^3\right)\right)$$

$$+10\sum_{cyc}\left(x^2y^2\left(\sum_{cyc}x-z\right)\right)+19xyz\sum_{cyc}x^2+30xyz\sum_{cyc}xy$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^5\overset{?}{\geq}6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3\right)-6xyz\left(\sum_{cyc}x^2\right)+10\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)$$

$$-10xyz\left(\sum_{cyc}xy\right)+19xyz\sum_{cyc}x^2+30xyz\sum_{cyc}xy$$

$$=6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3\right)+10\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)+13xyz\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)$$

$$-6xyz\left(\sum_{cyc}xy\right)$$

$$\Leftrightarrow\boxed{\left(\sum_{cyc}x\right)^5\underset{(*)}{\overset{?}{\geq}}6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3\right)+10\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)+13xyz\left(\sum_{cyc}x\right)^2-6xyz\left(\sum_{cyc}xy\right)}$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ **and** $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ *form* **sides of** a **triangle with semiperimeter, circumradius and inradius** $=s,R,r$ (*say*) **yielding** $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\rightarrow(1)\Rightarrow x=s-X, y=s-Y,$

$z=s-Z$ **and such substitutions** $\Rightarrow\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)$

$$\Rightarrow\sum_{cyc}xy=4Rr+r^2\rightarrow(2)\text{ and }\sum_{cyc}x^3=\left(\sum_{cyc}x\right)^3-3\prod_{cyc}(y+z)$$

$$\overset{\text{via }(1)}{=}s^3-3XYZ=s^3-12Rrs\Rightarrow\sum_{cyc}x^3=s^3-12Rrs\rightarrow(3)\text{ and also,}$$

$$\sum_{cyc}x^2y^2=\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)\overset{\text{via }(1)\text{ and }(2)}{=}\left(4Rr+r^2\right)^2$$

$$-2\left(\prod_{cyc}(s-X)\right).s=\left(4Rr+r^2\right)^2-2r^2s.s\Rightarrow\sum_{cyc}x^2y^2=r^2\left((4R+r)^2-2s^2\right)$$

$$\rightarrow (4)\ \therefore \text{ via } (1),(2),(3),(4),(*)\Leftrightarrow s^5\ge 6\left(4Rr+r^2\right)\left(s^3-12Rrs\right)$$

$$+10sr^2\left((4R+r)^2-2s^2\right)+13\left(\prod_{cyc}(s-X)\right)s^2-6r^2s\left(4Rr+r^2\right)$$

$$\Leftrightarrow s^4\ge 6\left(4Rr+r^2\right)\left(s^2-12Rr\right)+10r^2\left((4R+r)^2-2s^2\right)+13sr^2.s$$

$$-6r^2\left(4Rr+r^2\right)\Leftrightarrow s^4-\left(24Rr-r^2\right)s^2+4r^2\left(32R^2+4Rr-r^2\right)\ge 0$$

$$\Leftrightarrow s^2\ge\frac{24Rr-r^2+\sqrt{(24Rr-r^2)^2-16r^2(32R^2+4Rr-r^2)}}{2}$$

$$\Leftrightarrow \boxed{s^2\overset{(**)}{\ge}\frac{24Rr-r^2+r*\sqrt{64R^2-112Rr+17r^2}}{2}}$$

$$\text{Now, }2s^2\overset{\text{Rouche}}{\ge}2\left(2R^2+10Rr-r^2-2(R-2r)*\sqrt{R^2-2Rr}\right)\overset{?}{\ge}24Rr-r^2$$

$$+r*\sqrt{64R^2-112Rr+17r^2}$$

$$\Leftrightarrow \boxed{4R^2-4Rr-r^2-4(R-2r)*\sqrt{R^2-2Rr}\ \underbrace{\overset{?}{\ge}}_{(***)}\ r*\sqrt{64R^2-112Rr+17r^2}}$$

$$\because\left(4R^2-4Rr-r^2\right)^2-16\left(R^2-2Rr\right)(R-2r)^2$$

$$=r\left(64R^3-184R^2r+136Rr^2+r^3\right)$$

$$=(R-2r)\left(36R^2+28R(R-2r)+24r^2\right)+49r^3\overset{\text{Euler}}{\ge}49r^3>0$$

$$\therefore 4R^2-4Rr-r^2>4(R-2r)*\sqrt{R^2-2Rr}$$

$$\Rightarrow 4R^2-4Rr-r^2-4(R-2r)*\sqrt{R^2-2Rr}>0\Rightarrow(***)\Leftrightarrow$$

$$\left(4R^2-4Rr-r^2-4(R-2r)*\sqrt{R^2-2Rr}\right)^2\ge r^2\left(64R^2-112Rr+17r^2\right)$$

$$\Leftrightarrow\left(4R^2-4Rr-r^2\right)^2+16\left(R^2-2Rr\right)(R-2r)^2-r^2\left(64R^2-112Rr+17r^2\right)$$

$$\ge 8(R-2r)*\sqrt{R^2-2Rr}*\left(4R^2-4Rr-r^2\right)$$

$$\Leftrightarrow\boxed{(R-2r)\left(4R^3-8R^2r+Rr^2+r^3\right)\ge(R-2r)*\sqrt{R^2-2Rr}*\left(4R^2-4Rr-r^2\right)}$$

$$\Leftrightarrow 4R^3-8R^2r+Rr^2+r^3>\sqrt{R^2-2Rr}*\left(4R^2-4Rr-r^2\right)\ \left(\because R-2r\overset{\text{Euler}}{\ge}0\right)$$

$$\Leftrightarrow\left(4R^3-8R^2r+Rr^2+r^3\right)^2>\left(R^2-2Rr\right)\left(4R^2-4Rr-r^2\right)^2\Leftrightarrow\boxed{r^5(4R+r)>0}$$

$$\rightarrow \text{true}\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}\ \therefore\ \forall\, x,y,z>0,$$

$$\sum_{cyc}\frac{x}{y\left(6(x^3+y^3)+10xy(x+y)+xz(19x+30y)\right)}\ge\frac{1}{\left(\sum_{cyc}x\right)^3}$$

$$\Rightarrow\sum_{cyc}\frac{r_a}{r_b\left(6(r_a^3+r_b^3)+10r_ar_b(r_a+r_b)+r_ar_c(19r_a+30r_b)\right)}\ge\frac{1}{\left(\sum_{cyc}r_a\right)^3}$$

$$=\frac{1}{(4R+r)^3}\overset{\text{Euler}}{\ge}\frac{1}{\left(4R+\frac{R}{2}\right)^3}=\left(\frac{2}{9R}\right)^3,\text{"="}\ \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$LHS \overset{CBS}{\geq} \frac{\left(\sum_{cyc}\sqrt{\frac{r_a}{r_b}}\right)^2}{\sum_{cyc}[6(r_a{}^3+r_b{}^3)+10r_ar_b(r_a+r_b)+r_ar_c(19r_a+30r_b)]}$$

$$\overset{AM-GM}{\geq} \frac{3^2}{12\sum_{cyc}r_a{}^3+10\sum_{cyc}r_a{}^2r_b+29\sum_{cyc}r_ar_b{}^2+90r_ar_br_c}$$

$$\overset{AM-GM}{\geq} \frac{9}{12\sum_{cyc}r_a{}^3+29\sum_{cyc}r_ar_b(r_a+r_b)+33r_ar_br_c}$$

$$= \frac{9}{12\left(\sum_{cyc}r_a\right)^3-7\sum_{cyc}r_a\cdot\sum_{cyc}r_br_c-18r_ar_br_c}$$

$$= \frac{9}{12(4R+r)^3-7(4R+r)s^2-18s^2r}$$

$$\overset{Gerretsen}{\geq} \frac{9}{12(4R+r)^3-(28R+25r)(16Rr-5r^2)} \overset{?}{\geq} \left(\frac{2}{9R}\right)^3$$

$$\Leftrightarrow 417R^3-1024R^2r+928Rr^2-1096r^3 \geq 0$$

$$\Leftrightarrow\ (R-2r)(417R^2-190Rr+548r^2) \geq 0$$

which is true by Euler's inequality $R \geq 2r$. Equality holds iff ΔABC is equilateral.

1343. In any Δ ABC, the following relationship holds :

$$\frac{(r_aw_b)^5}{r_a^5+w_b^5}+\frac{(r_bw_c)^5}{r_b^5+w_c^5}+\frac{(r_cw_a)^5}{r_c^5+w_a^5} \leq \left(\frac{3}{2}\right)^6\frac{(81R^5-2560r^5)^2}{32r^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$w_a = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)} \overset{A-G}{\leq} \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-b)(s-c)}} \overset{G-H}{\leq}$$

$$\frac{rs(s-a)(s-b+s-c)}{2(s-a)(s-b)(s-c)} = \frac{rsa(s-a)}{2r^2s} = \frac{a(s-a)}{2r}$$

$$\Rightarrow w_a^5 \leq s^2(s-a)^2\cdot\frac{a(s-a)}{2r} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}w_a^5 \leq \frac{s^2}{2r}\cdot\sum_{cyc}a(s-a)^3 = \frac{s^2}{2r}\cdot\sum_{cyc}a(s^3-3s^2a+3sa^2-a^3)$$

$$= \frac{s^2}{2r}\cdot\begin{pmatrix} s^3(2s)-6s^2(s^2-4Rr-r^2)+6s^2(s^2-6Rr-3r^2) \\ +16r^2s^2-2\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)\end{pmatrix}$$

$$= \frac{s^2}{2r}\cdot\left(4Rrs^2-2r^2(4R+r)^2\right) \overset{Gerretsen}{\leq}$$

$$\frac{s^2}{2r}\cdot\left(4Rr(4R^2+4Rr+3r^2)-2r^2(4R+r)^2\right)$$

$$\Rightarrow \sum_{cyc} w_a^5 \overset{(i)}{\leq} s^2\left(8R^3 - 8R^2r - 2Rr^2 - r^3\right)$$

$$r_a \leq \frac{a^2}{4r} \Rightarrow r_a^5 \leq s^4 \tan^4\frac{A}{2}.\frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{4r} = \frac{4R^2s^4}{r}\left(\tan^2\frac{A}{2}\right)\left(1-\cos^2\frac{A}{2}\right)^2$$

$$= \frac{4R^2s^4}{r}\left(\tan^2\frac{A}{2}\right)\left(1+\cos^4\frac{A}{2} - 2\cos^2\frac{A}{2}\right) = \frac{4R^2s^4}{r}\left(\frac{r_a^2}{s^2} + \frac{\sin^2 A}{4} - (1-\cos A)\right)$$

$$\Rightarrow r_a^5 \leq \frac{4R^2s^4}{r}\left(\frac{r_a^2}{s^2} + \frac{a^2}{16R^2} - 1 + \cos A\right) \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} r_a^5 \leq \frac{4R^2s^4}{r}\left(\frac{(4R+r)^2 - 2s^2}{s^2} + \frac{\sum_{cyc} a^2}{16R^2} - 3 + 1 + \frac{r}{R}\right)$$

$$\overset{\text{Leibnitz}}{\leq} \frac{4R^2s^4}{r}\left(\frac{(4R+r)^2 - 4s^2}{s^2} + \frac{9}{16} + \frac{r}{R}\right)$$

$$= \frac{Rs^2}{4r}.\left(16R(4R+r)^2 - (55R-16r)s^2\right) \overset{\text{Gerretsen}}{\leq}$$

$$\frac{Rs^2}{4r}.\left(16R(4R+r)^2 - (55R-16r)\left(16Rr-5r^2\right)\right)$$

$$\Rightarrow \sum_{cyc} r_a^5 \overset{(ii)}{\leq} \frac{Rs^2}{4r}.\left(256R^3 - 752R^2r + 547Rr^2 - 80r^3\right)$$

$$\because 256R^3 - 752R^2r + 547Rr^2 - 80r^3$$

$$= (R-2r)\left(136R^2 + 120R(R-2r) + 67r^2\right) + 54r^3 \overset{\text{Euler}}{\geq} 54r^3 > 0 \text{ and}$$

$$8R^3 - 8R^2r - 2Rr^2 - r^3 = (R-2r)\left(8R^2 + 8Rr + 14r^2\right) + 27r^3 \overset{\text{Euler}}{\geq} 27r^3 > 0$$

$$\therefore \frac{(r_a w_b)^5}{r_a^5 + w_b^5} + \frac{(r_b w_c)^5}{r_b^5 + w_c^5} + \frac{(r_c w_a)^5}{r_c^5 + w_a^5} \overset{A-H}{\leq} \frac{1}{4}\left(\sum_{cyc} w_a^5 + \sum_{cyc} r_a^5\right)$$

$$\overset{\text{via (i),(ii)}}{\leq} \frac{s^2}{4}\left(8R^3 - 8R^2r - 2Rr^2 - r^3\right) + \frac{Rs^2}{16r}.\left(256R^3 - 752R^2r + 547Rr^2 - 80r^3\right)$$

$$\overset{\text{Mitrinovic}}{\leq} \frac{27R^2}{16}\left(8R^3 - 8R^2r - 2Rr^2 - r^3\right)$$

$$+ \frac{27R^3}{64r}.\left(256R^3 - 752R^2r + 547Rr^2 - 80r^3\right) \overset{?}{\leq} \left(\frac{3}{2}\right)^6 \frac{\left(81R^5 - 2560r^5\right)^2}{32r^5}$$

$$\Leftrightarrow 27\left(81R^5 - 2560r^5\right)^2 \overset{?}{\geq} 32r^4\left(\begin{array}{c} 4R^2r\left(8R^3 - 8R^2r - 2Rr^2 - r^3\right) \\ +R^3\left(256R^3 - 752R^2r + 547Rr^2 - 80r^3\right)\end{array}\right)$$

$$\Leftrightarrow 177147t^{10} - 8192t^6 - 11174400t^5 - 16480t^4 + 2816t^3 + 128t^2$$

$$+176947200 \overset{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(\begin{pmatrix}(t-2)*\\ 177147t^8+708588t^7+2125764t^6\\ +5668704t^5+14163568t^4+22805056t^3\\ +34549472t^2+46980480t+49724160\end{pmatrix}\\ +10974720\right)\overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \frac{(r_a w_b)^5}{r_a^5+w_b^5}+\frac{(r_b w_c)^5}{r_b^5+w_c^5}+\frac{(r_c w_a)^5}{r_c^5+w_a^5}$$

$$\leq \left(\frac{3}{2}\right)^6\frac{(81R^5-2560r^5)^2}{32r^5}\ \forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1344. Let ABC be an acute triangle and let H, I be the orthocenter, incenter of ΔABC respectively and $A_1B_1C_1$ the orthic triangle. Then

$$\frac{HA_1.HB_1.HC_1}{IA_1.IB_1.IC_1}.\frac{HA.HB.HC}{IA.IB.IC}\leq\frac{R^5}{32r^5}$$

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $HA=2R\cos A$ and $HA_1=2R\cos B\cos C$ (and analogs), *and since*

$$\cos A\cos B\cos C \overset{AM-GM}{\leq} \left(\frac{\cos A+\cos B+\cos C}{3}\right)^3 \overset{Jensen}{\leq} \cos^3\frac{\pi}{3}=\frac{1}{8},\ \text{then}$$

$$HA.HB.HC=8R^3\cos A\cos B\cos C\leq R^3 \text{ and}$$

$$HA_1.HB_1.HC_1=8R^3(\cos A\cos B\cos C)^2=\frac{R^3}{8}.$$

Also, we have $IA.IB.IC=\dfrac{r}{\sin\frac{A}{2}}.\dfrac{r}{\sin\frac{B}{2}}.\dfrac{r}{\sin\frac{C}{2}}=\dfrac{r^3}{\frac{r}{4R}}=4Rr^2.$

Now, since $IA_1\geq distance(I,(BC))=r$ (and analogs), then $IA_1.IB_1.IC_1\geq r^3$.

Therefore

$$\frac{HA_1.HB_1.HC_1}{IA_1.IB_1.IC_1}.\frac{HA.HB.HC}{IA.IB.IC}\leq\frac{\frac{R^3}{8}}{r^3}.\frac{R^3}{4Rr^2}=\frac{R^5}{32r^5}.$$

Equality holds iff ΔABC is equilateral.

1345.

In any ΔABC and $n, m\in\mathbb{N}$ such that $n\geq m-2$ the following relationship holds :

$$\frac{r_a^{\,n}(r_a^{\,2}+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^{\,n}(r_b^{\,2}+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^{\,n}(r_c^{\,2}+r_ar_b)}{(r_a+r_b)^m}\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{r_a{}^n(r_a{}^2+r_br_c)}{(r_b+r_c)^m}=\frac{r_a{}^n}{(r_b+r_c)^m}(r_a-r_b)(r_a-r_c)+\frac{r_a{}^{n+1}}{(r_b+r_c)^{m-1}}\ \text{(and analogs)}$$

WLOG, we may assume that $r_a\ge r_b\ge r_c$. **We have**

$$\frac{r_a{}^n}{(r_b+r_c)^m}\ge\frac{r_b{}^n}{(r_c+r_a)^m}\ge\frac{r_c{}^n}{(r_a+r_b)^m},$$

then by the Generalized Schur inequality, we have

$$\sum_{cyc}\frac{r_a{}^n}{(r_b+r_c)^m}(r_a-r_b)(r_a-r_c)\ge 0.$$

Therefore

$$\sum_{cyc}\frac{r_a{}^n(r_a{}^2+r_br_c)}{(r_b+r_c)^m}\ge\sum_{cyc}\frac{r_a{}^{n+1}}{(r_b+r_c)^{m-1}}\overset{Chebyshev}{\ge}\frac{1}{3}\cdot\sum_{cyc}r_a{}^{n+1}\cdot\sum_{cyc}\frac{1}{(r_b+r_c)^{m-1}}$$

$$\overset{H\ddot{o}lder}{\ge}\frac{1}{3}\cdot\frac{(r_a+r_b+r_c)^{n+1}}{3^n}\cdot\frac{3^m}{2^{m-1}(r_a+r_b+r_c)^{m-1}}=\frac{(4R+r)^{n-m+2}}{3^{n+1-m}.2^{m-1}}$$

$$\overset{Euler}{\ge}\frac{(9r)^{n-m+2}}{3^{n+1-m}.2^{m-1}}=\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}.$$

as desired. Equality holds if and only if ΔABC is equilateral.

1346. In any acute Δ ABC, the following identity occurs :

$$\sin\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)-\sin\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)$$

$$=\frac{(a^2+b^2+c^2-6R^2)\cos(\hat{B}-\hat{C})}{R}$$

Proposed by Radu Diaconu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}AH\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)=R\sum_{cyc}2\cos\hat{A}\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)$$

$$=R\sum_{cyc}\left(\cos\left(\hat{A}+\hat{B}-\hat{C}-\frac{\pi}{4}\right)+\cos\left(\hat{A}-\hat{B}+\hat{C}+\frac{\pi}{4}\right)\right)$$

$$=R\sum_{cyc}\left(\cos\left(\frac{3\pi}{4}-2\hat{C}\right)+\cos\left(\frac{5\pi}{4}-2\hat{B}\right)\right)$$

$$=R\sum_{cyc}\left(-\frac{1}{\sqrt{2}}\cos\widehat{2C}+\frac{1}{\sqrt{2}}\sin\widehat{2C}-\frac{1}{\sqrt{2}}\cos\widehat{2B}-\frac{1}{\sqrt{2}}\sin\widehat{2B}\right)$$

$$=\frac{R}{\sqrt{2}}\left(\sum_{cyc}\sin 2\widehat{C}-\sum_{cyc}\sin 2\widehat{B}-\sum_{cyc}\cos 2\widehat{C}-\sum_{cyc}\cos 2\widehat{B}\right)$$

$$=-R*\sqrt{2}\left(\sum_{cyc}\cos 2\widehat{A}\right)\ and\ \because \sin\left(\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)$$

$$=\frac{1}{\sqrt{2}}\left(\cos(\widehat{B}-\widehat{C})+\sin(\widehat{B}-\widehat{C})\right)$$

$$\therefore \boxed{\sin\left(\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\widehat{B}-\widehat{C}-\frac{\pi}{4}\right)\overset{(*)}{=}R\left(\sum_{cyc}\cos 2\widehat{A}\right)\left(-\cos(\widehat{B}-\widehat{C})-\sin(\widehat{B}-\widehat{C})\right)}$$

$$\text{Also,}\sum_{cyc}AH\cos\left(\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)=R\sum_{cyc}2\cos\widehat{A}\cos\left(\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)$$

$$=R\sum_{cyc}\left(\cos\left(\widehat{A}+\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)+\cos\left(\widehat{A}-\widehat{B}+\widehat{C}-\frac{\pi}{4}\right)\right)$$

$$=R\sum_{cyc}\left(\cos\left(\frac{5\pi}{4}-2\widehat{C}\right)+\cos\left(\frac{3\pi}{4}-2\widehat{B}\right)\right)$$

$$=R\sum_{cyc}\left(-\frac{1}{\sqrt{2}}\cos 2\widehat{C}-\frac{1}{\sqrt{2}}\sin 2\widehat{C}-\frac{1}{\sqrt{2}}\cos 2\widehat{B}+\frac{1}{\sqrt{2}}\sin 2\widehat{B}\right)$$

$$=\frac{R}{\sqrt{2}}\left(-\sum_{cyc}\cos 2\widehat{C}-\sum_{cyc}\sin 2\widehat{C}-\sum_{cyc}\cos 2\widehat{B}+\sum_{cyc}\sin 2\widehat{B}\right)$$

$$=-R*\sqrt{2}\left(\sum_{cyc}\cos 2\widehat{A}\right)\ and\ \because -\sin\left(\widehat{B}-\widehat{C}-\frac{\pi}{4}\right)$$

$$=\frac{1}{\sqrt{2}}\left(\cos(\widehat{B}-\widehat{C})-\sin(\widehat{B}-\widehat{C})\right)$$

$$\therefore \boxed{-\sin\left(\widehat{B}-\widehat{C}-\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)\overset{(**)}{=}R\left(\sum_{cyc}\cos 2\widehat{A}\right)\left(\sin(\widehat{B}-\widehat{C})-\cos(\widehat{B}-\widehat{C})\right)}$$

$$\therefore (*),(**)\Rightarrow \sin\left(\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\widehat{B}-\widehat{C}-\frac{\pi}{4}\right)$$

$$-\sin\left(\widehat{B}-\widehat{C}-\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\widehat{B}-\widehat{C}+\frac{\pi}{4}\right)=-2R\left(-1-4\prod_{cyc}\cos\widehat{A}\right)\cos(\widehat{B}-\widehat{C})$$

$$=2R\cos(\widehat{B}-\widehat{C})*\left(1+\frac{s^2-4R^2-4Rr-r^2}{R^2}\right)$$

$$=2R\cos(\widehat{B}-\widehat{C})*\left(\frac{s^2-4Rr-r^2-3R^2}{R^2}\right)$$

$$=\frac{\cos(\widehat{B}-\widehat{C})}{R}*\left(2(s^2-4Rr-r^2)-6R^2\right)$$

$$= \frac{(a^2+b^2+c^2-6R^2)\cos(\hat{B}-\hat{C})}{R} \text{ (QED)}$$

1347. In any Δ ABC, Δ A′B′C′, the following relationship holds :

$$\min\left\{\sum_{cyc}\frac{m_a'}{m_b'+m_c'}, \sum_{cyc}\frac{a}{b+c}\right\} + \frac{R'R^2}{r'r^2} \geq 8 + \max\left\{\sum_{cyc}\frac{m_a}{m_b+m_c}, \sum_{cyc}\frac{a'}{b'+c'}\right\}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum m_a m_b \overset{\text{Tereshin}}{\geq} \sum \frac{(b^2+c^2)(c^2+a^2)}{16R^2} = \frac{3\sum_{cyc}a^2b^2+\sum_{cyc}a^4}{16R^2}$$

$$\geq \frac{3\sum_{cyc}a^2b^2+\sum_{cyc}a^2b^2}{16R^2} = \sum_{cyc}\frac{a^2b^2}{4R^2} \Rightarrow \frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}$$

$$\leq \frac{4R^2 * \frac{3}{2}(s^2-4Rr-r^2)}{\sum_{cyc}a^2b^2} \overset{?}{\leq} \frac{R}{2r} \Leftrightarrow \sum_{cyc}a^2b^2 \overset{?}{\geq} 12Rr(s^2-4Rr-r^2)$$

$$\Leftrightarrow (s^2+4Rr+r^2)^2 - 16Rrs^2 \overset{?}{\geq} 12Rr(s^2-4Rr-r^2)$$

$$\Leftrightarrow s^4 - (20Rr-2r^2)s^2 + r^2(64R^2+20Rr+r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow s^2 \geq \frac{20Rr-2r^2+\sqrt{(20Rr-2r^2)^2-4r^2(64R^2+20Rr+r^2)}}{2}$$

$$\Leftrightarrow s^2 \geq \frac{20Rr-2r^2+4r*\sqrt{9R^2-10Rr}}{2} \Leftrightarrow s^2 \overset{(*)}{\geq} 10Rr-r^2+2r*\sqrt{9R^2-10Rr}$$

$$\text{Now, } s^2 \overset{\text{Rouche}}{\geq} 2R^2+10Rr-r^2-2(R-2r)*\sqrt{R^2-2Rr} \overset{?}{\geq} 10Rr-r^2+$$
$$2r*\sqrt{9R^2-10Rr} \Leftrightarrow R^2-(R-2r)*\sqrt{R^2-2Rr} \underset{(**)}{\overset{?}{\geq}} r*\sqrt{9R^2-10Rr}$$

$$\because R^4-(R^2-2Rr)(R-2r)^2 = 2Rr(3R^2-6Rr+4r^2) \overset{\text{Euler}}{\geq} 8Rr^3 > 0$$

$$\Rightarrow R^2 > (R-2r)*\sqrt{R^2-2Rr} \Rightarrow R^2-(R-2r)*\sqrt{R^2-2Rr} > 0$$

$$\Rightarrow (**) \Leftrightarrow \left(R^2-(R-2r)*\sqrt{R^2-2Rr}\right)^2 \geq r^2(9R^2-10Rr)$$

$$\Leftrightarrow R^4+(R^2-2Rr)(R-2r)^2-r^2(9R^2-10Rr) \geq 2R^2*(R-2r)*\sqrt{R^2-2Rr}$$

$$\Leftrightarrow R(R-2r)(2R^2-2Rr-r^2) \geq 2R^2*(R-2r)*\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 2R^2-2Rr-r^2 > 2R*\sqrt{R^2-2Rr} \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)$$

$$\Leftrightarrow (2R^2-2Rr-r^2)^2 - 4R^2(R^2-2Rr) > 0$$

$$\Leftrightarrow r^3(4R+r) > 0 \to true \Rightarrow (**) \Rightarrow (*) \text{ is true} \Rightarrow \frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a} \leq \frac{R}{2r}$$

$$\Rightarrow \frac{3}{2}+\frac{R'R^2}{r'r^2} \overset{\text{Euler}}{\geq} \frac{3}{2}+8*\frac{R^2}{4r^2} \geq \frac{3}{2}+8\left(\frac{\sum_{cyc} m_a^2}{\sum_{cyc} m_a m_b}\right)^2 \therefore \text{in order to prove :}$$

$$\frac{3}{2}+\frac{R'R^2}{r'r^2} \geq 8+\sum_{cyc}\frac{m_a}{m_b+m_c}, \text{it suffices to prove :}$$

$$\frac{3}{2}+8\left(\frac{\sum_{cyc} m_a^2}{\sum_{cyc} m_a m_b}\right)^2 \overset{(\blacksquare)}{\geq} 8+\sum_{cyc}\frac{m_a}{m_b+m_c}$$

$$\text{Now, } \frac{3}{2}+8\left(\frac{\sum_{cyc} a^2}{\sum_{cyc} ab}\right)^2 \overset{?}{\geq} 8+\sum_{cyc}\frac{a}{b+c}$$

$$\Leftrightarrow \frac{8}{(s^2+4Rr+r^2)^2}\left(\sum_{cyc} a^2+\sum_{cyc} ab\right)\left(\sum_{cyc} a^2-\sum_{cyc} ab\right) \overset{?}{\geq} 2s\sum_{cyc}\frac{1}{b+c}-\frac{9}{2}$$

$$\Leftrightarrow \frac{8(3s^2-4Rr-r^2)(s^2-12Rr-3r^2)}{(s^2+4Rr+r^2)^2} \overset{?}{\geq} \frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{9}{2}$$

$$= \frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)} \Leftrightarrow 8(3s^2-4Rr-r^2)(2s^2-24Rr-6r^2)(s^2+2Rr+r^2)$$

$$\underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}} (s^2-10Rr-7r^2)(s^2+4Rr+r^2)^2$$

$$2s^2-24Rr-6r^2 \overset{?}{\geq} s^2-10Rr-7r^2 \Leftrightarrow s^2-16Rr+5r^2+2r(R-2r) \overset{?}{\geq} 0$$

$\rightarrow$ **true via Gerretsen and Euler $\therefore$ in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :**

$$8(3s^2-4Rr-r^2)(s^2+2Rr+r^2) > (s^2+4Rr+r^2)^2$$

$$\Leftrightarrow 23s^4+(8Rr+14r^2)s^2-r^2(80R^2+56Rr+9r^2) \overset{(\blacksquare\blacksquare\blacksquare)}{>} 0$$

$$\text{Now, LHS of } (\blacksquare\blacksquare\blacksquare) \overset{\text{Gerretsen}}{\geq} \left(23(16Rr-5r^2)+(8Rr+14r^2)\right)(16Rr-5r^2)$$

$$-r^2(80R^2+56Rr+9r^2) = r^2\left((R-2r)(5936R+8320r)+17136r^2\right) \overset{\text{Euler}}{\geq}$$

$$17136r^4 > 0 \Rightarrow (\blacksquare\blacksquare\blacksquare) \Rightarrow (\blacksquare\blacksquare) \text{ is true} \Rightarrow \frac{3}{2}+8\left(\frac{\sum_{cyc} a^2}{\sum_{cyc} ab}\right)^2 \geq 8+\sum_{cyc}\frac{a}{b+c}$$

and implementing it on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, we arrive at :

$$\frac{3}{2}+8\left(\frac{\sum_{cyc} m_a^2}{\sum_{cyc} m_a m_b}\right)^2 \geq 8+\sum_{cyc}\frac{m_a}{m_b+m_c} \Rightarrow (\blacksquare) \text{ is true}$$

$$\Rightarrow \boxed{\frac{3}{2}+\frac{R'R^2}{r'r^2} \overset{(?)}{\geq} 8+\sum_{cyc}\frac{m_a}{m_b+m_c}}$$

$$\text{Again, } \frac{3}{2}+\frac{4R}{r} \overset{?}{\geq} 8+\sum_{cyc}\frac{a}{b+c} \Leftrightarrow \frac{4R}{r}-8 \overset{?}{\geq} 2s\sum_{cyc}\frac{1}{b+c}-\frac{9}{2}$$

$$= \frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{9}{2} = \frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)} \Leftrightarrow \frac{4(R-2r)}{r} \overset{?}{\geq} \frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)}$$

$$\Leftrightarrow (8R-17r)s^2+r(16R^2-14Rr-9r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow (8R-16r)s^2 - rs^2 + r\left(16R^2 - 14Rr - 9r^2\right) \underset{(\bullet)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\geq} (8R-16r)\left(16Rr - 5r^2\right) - r\left(4R^2 + 4Rr + 3r^2\right)$$

$$+r\left(16R^2 - 14Rr - 9r^2\right) \overset{?}{\geq} 0 \Leftrightarrow 70R^2 - 157Rr + 34r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(70R-17r) \overset{?}{\geq} 0 \to \text{true via Euler} \Rightarrow (\bullet) \text{ is true} \Rightarrow \frac{3}{2} + \frac{4R}{r}$$

$$\geq 8 + \sum_{cyc} \frac{a}{b+c} \therefore \frac{3}{2} + \frac{R'R^2}{r'r^2} \overset{\text{Euler}}{\geq} \frac{3}{2} + \frac{4R'}{r'} \geq 8 + \sum_{cyc} \frac{a'}{b'+c'}$$

$$\therefore \boxed{\frac{3}{2} + \frac{R'R^2}{r'r^2} \overset{(\square\square)}{\geq} 8 + \sum_{cyc} \frac{a'}{b'+c'}}$$

$$\therefore \min\left\{\sum_{cyc} \frac{m_a{}'}{m_b{}' + m_c{}'}, \sum_{cyc} \frac{a}{b+c}\right\} + \frac{R'R^2}{r'r^2} \overset{\text{Nesbitt}}{\geq} \frac{3}{2} + \frac{R'R^2}{r'r^2} \overset{\text{via } (\square),(\square\square)}{\geq}$$

$$8 + \max\left\{\sum_{cyc} \frac{m_a}{m_b + m_c}, \sum_{cyc} \frac{a'}{b'+c'}\right\}, '' ='' \text{ iff } \Delta\, ABC, \Delta\, A'B'C' \text{ are both equilateral (QED)}$$

1348. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \frac{2 + \sqrt{3}\tan\frac{B}{2}}{1 + 3\tan^2\frac{A}{2}} \geq \frac{9}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1+3x^2} \overset{?}{\geq} 1 - \frac{\sqrt{3}x}{2} \Leftrightarrow \frac{\sqrt{3}x}{2} \overset{?}{\geq} 1 - \frac{1}{1+3x^2} = \frac{3x^2}{1+3x^2} \Leftrightarrow 1 + 3x^2 \overset{?}{\geq} 2\sqrt{3}x$$

$$\Leftrightarrow \left(1 - \sqrt{3}x\right)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \frac{1}{1+3x^2} \geq 1 - \frac{\sqrt{3}x}{2}$$

$$\Rightarrow \frac{1}{1 + 3\tan^2\frac{A}{2}} - \left(1 - \frac{\sqrt{3}\tan\frac{A}{2}}{2}\right) \geq 0$$

$$\Rightarrow \left(2 + \sqrt{3}\tan\frac{B}{2}\right)\left(\frac{1}{1 + 3\tan^2\frac{A}{2}} - \left(1 - \frac{\sqrt{3}\tan\frac{A}{2}}{2}\right)\right) \geq 0$$

$$\Rightarrow \frac{2 + \sqrt{3}\tan\frac{B}{2}}{1 + 3\tan^2\frac{A}{2}} \geq \left(2 + \sqrt{3}\tan\frac{B}{2}\right)\left(1 - \frac{\sqrt{3}\tan\frac{A}{2}}{2}\right)$$

$$= 2+\sqrt{3}\tan\frac{B}{2}-\sqrt{3}\tan\frac{A}{2}-\frac{3}{2}\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\therefore \frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq 2+\sqrt{3}\tan\frac{B}{2}-\sqrt{3}\tan\frac{A}{2}-\frac{3r_ar_b}{2s^2} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq 6+\sqrt{3}\sum_{cyc}\tan\frac{B}{2}-\sqrt{3}\sum_{cyc}\tan\frac{A}{2}-\frac{3}{2s^2}\sum_{cyc}r_ar_b = 6-\frac{3s^2}{2s^2}=\frac{9}{2}$$

$$\therefore \text{in any } \Delta\, ABC, \sum_{cyc}\frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq \frac{9}{2}, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1349.

In any Δ ABC with I $\rightarrow$ incenter, the following relationship holds :

$$\sqrt{\frac{2R}{r}}\cdot\sum_{cyc}\frac{n_a}{\sqrt{(b-c)^2+4r^2}} \geq \frac{1}{\sqrt{2}}\cdot\sum_{cyc}\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{1}{r}\sum_{cyc}AI$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \overset{?}{\geq} 32Rr\cos^2\frac{A}{2} \overset{\text{via (i)}}{=} 8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0 \Leftrightarrow (b+c-2a)^2\overset{?}{\geq}0 \rightarrow \text{true}$$

$$\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\Rightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\geq\sqrt{32Rr}.\cos\frac{A}{2}\Rightarrow\cos\frac{B-C}{2}\overset{(ii)}{\geq}\sqrt{\frac{2r}{R}}$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$

$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}\Rightarrow a^2n_a^2=a^2s^2-sa(a^2-(b-c)^2)$$

$$\Rightarrow a^2n_a^2=a^2s^2-sa^3+sa(b-c)^2\overset{?}{\geq}4r^2s^2+s^2(b-c)^2$$

$$\Leftrightarrow a^2s^2-sa^3+sa(b-c)^2\overset{?}{\geq}s(s-a)(a^2-(b-c)^2)+s^2(b-c)^2$$

$$=a^2s^2-sa^3-s(s-a)(b-c)^2+s^2(b-c)^2$$

$$=a^2s^2-sa^3-s^2(b-c)^2+sa(b-c)^2+s^2(b-c)^2$$

$$\Leftrightarrow a^2s^2 - sa^3 + sa(b-c)^2 \overset{?}{=} a^2s^2 - sa^3 + sa(b-c)^2 \to \text{true}$$

$$\therefore \frac{n_a^2}{(b-c)^2+4r^2} = \frac{s^2}{a^2} \Rightarrow \sqrt{\frac{2R}{r}}.\frac{n_a}{\sqrt{(b-c)^2+4r^2}} = \sqrt{\frac{R}{2r}}.\frac{2s}{a}$$

$$= \sqrt{\frac{R}{2r}}.\frac{4R.2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}} = \sqrt{\frac{R}{2r}}.\frac{\sin\frac{A}{2}+\cos\frac{B-C}{2}}{\sin\frac{A}{2}} = \sqrt{\frac{R}{2r}} + \sqrt{\frac{R}{2r}}.\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$$

$$\overset{\text{via (ii)}}{\geq} \sqrt{\frac{R}{2r}} + \sqrt{\frac{R}{2r}}.\frac{\sqrt{\frac{2r}{R}}}{\sin\frac{A}{2}} = \sqrt{\frac{R}{2r}} + \frac{AI}{r} \geq \sqrt{\frac{\frac{m_b}{h_c}+\frac{m_c}{h_b}}{2}} + \frac{AI}{r}$$

$$\left(\because \frac{R}{r} \geq \frac{m_b}{h_c}+\frac{m_c}{h_b}; \text{reference: article titled ''NewTriangle Inequalities With Brocard's Angle'' by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 - 7, published at : www.ssmrmh.ro}\right)$$

$$\therefore \sqrt{\frac{2R}{r}}.\frac{n_a}{\sqrt{(b-c)^2+4r^2}} \geq \frac{1}{\sqrt{2}}.\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}} + \frac{AI}{r} \text{ and analogs}$$

$$\Rightarrow \sqrt{\frac{2R}{r}}.\sum_{cyc}\frac{n_a}{\sqrt{(b-c)^2+4r^2}} \geq \frac{1}{\sqrt{2}}.\sum_{cyc}\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}} + \frac{1}{r}\sum_{cyc}AI,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We prove the result by using the following inequality (see, Bogdan Fuştei, Mohamed Amine Ben Ajiba, *New Triangle Inequalities With Brocard's Angle*, Lemma 12, 6 − 7, www.ssmrmh.ro)

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \frac{R}{r} \quad \text{(and analogs)}$$

Also, we have

$$\frac{AI}{r} = \frac{1}{\sin\frac{A}{2}} = \sqrt{\frac{bc}{(s-b)(s-c)}} = \sqrt{\frac{2Rsr.(b+c-a).a}{sr^2.a^2}} \overset{AM-GM}{\leq} \sqrt{\frac{R}{2r}}.\frac{(b+c-a)+a}{a}$$

$$= \sqrt{\frac{R}{2r}}.\frac{b+c}{a}.$$

$$(b-c)^2 + 4r^2 = (b-c)^2 + \frac{(s-a).4(s-b)(s-c)}{s}$$

$$= (b-c)^2 + \frac{(s-a)[a^2-(b-c)^2]}{s}$$

$$=\frac{a^2(s-a)+a(b-c)^2}{s}=\frac{a^2}{s^2}\left(s(s-a)+\frac{s(b-c)^2}{a}\right)=\left(\frac{an_a}{s}\right)^2.$$

Using these results, we have

$$\frac{1}{\sqrt{2}}\cdot\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{AI}{r}\le\sqrt{\frac{R}{2r}}+\sqrt{\frac{R}{2r}}\cdot\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\cdot\frac{s}{a}$$

$$=\sqrt{\frac{2R}{r}}\cdot\frac{n_a}{\sqrt{(b-c)^2+4r^2}}\quad\text{(and analogs)}$$

Therefore

$$\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{1}{r}\cdot\sum_{cyc}AI\le\sqrt{\frac{2R}{r}}\cdot\sum_{cyc}\frac{n_a}{\sqrt{(b-c)^2+4r^2}}.$$

Equality holds iff ΔABC is equilateral.

1350. **In ΔABC the following relationship holds:**

$$\frac{1}{m_a^4m_b}+\frac{1}{m_b^4m_c}+\frac{1}{m_c^4m_a}\ge\frac{32}{81(81R^5-2560r^5)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{1}{m_a^4m_b}=\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^4}{m_b}\overset{HOLDER}{\ge}\frac{\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^4}{9(m_a+m_b+m_c)}\ge$$

$$\overset{LEUENBERGER}{\ge}\frac{\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^4}{9(4R+r)}\overset{BERGSTROM}{\ge}\frac{\left(\frac{9}{m_a+m_b+m_c}\right)^4}{9(4R+r)}\ge$$

$$\overset{LEUENBERGER}{\ge}\frac{\left(\frac{9}{4R+r}\right)^4}{9(4R+r)}\overset{EULER}{\ge}\frac{\left(\frac{9}{4R+\frac{R}{2}}\right)^4}{9(4R+r)}=\frac{\left(\frac{2}{R}\right)^4}{9(4R+r)}=\frac{16}{9R^4(4R+r)}$$

Remains to prove:

$$\frac{16}{9R^4(4R+r)}\ge\frac{32}{81(81R^5-2560r^5)}$$

$$\frac{1}{R^4(4R+r)}\ge\frac{2}{9(81R^5-2560r^5)}$$

$$721R^5 - 2R^4r - 9 \cdot 2560r^5 \geq 0$$

$$721R^4(R-2r) + 2^5 \cdot 3^2 \cdot 5r(R^4 - 16r^4) \geq 0$$

$$(R-2r)\left(721R^4 + 2^5 \cdot 3^2 \cdot 5r(R+2r)(R^2+4r^2)\right) \geq 0$$

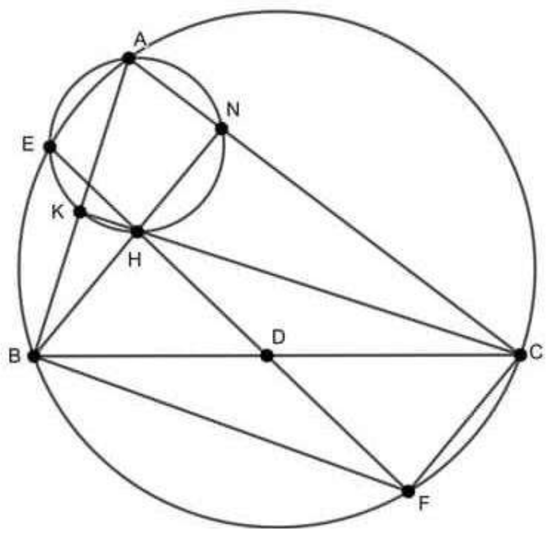
$$R - 2r \geq 0$$

$R \geq 2r$ (Euler)

Equality holds for: $a = b = c$.

1351. ***H*** **orthocenter of** ΔABC**,** $BN \perp AC, CK \perp AB, E, H, D, F$ **collinears**

Prove that: $BHCF$ **parallelogram**



Proposed by Eldeniz Hesenov-Georgia

Solution by Rajarshi Chakraborty-India

E, H, D, F are colinears

Also since H is the orthocentre, B, H, N are colinear and K, H, C are colinear

$$\angle EHK = \angle EAK = \angle EAB = \angle EFB \Rightarrow HC \parallel BF, \angle PEB = \angle ACB$$

$$\angle PEF = \angle ANB = 90^\circ$$

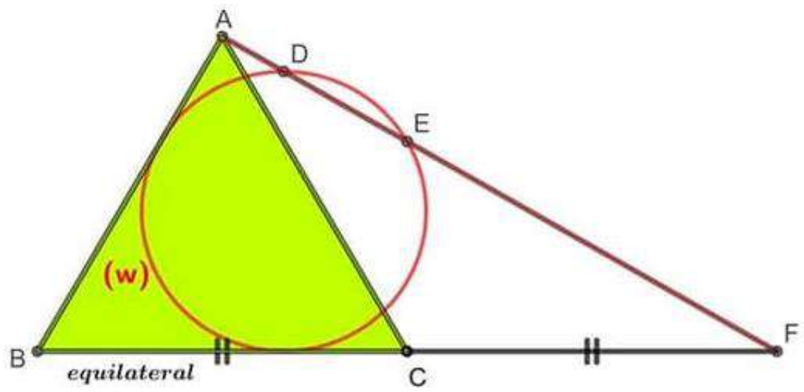
$$\therefore \angle BEF = 90^\circ - \angle ACB \Rightarrow \angle BCF = 90^\circ - \angle ACB \Rightarrow \angle BCF + \angle ACB = 90^\circ$$

$\Rightarrow \angle ACF = 90^\circ$.Also $\angle ANB = 90^\circ$, $\therefore BH \parallel FC$, $\therefore BHCF$ is a parallelogram

1352. (w) **– mixtilinear incircle of** ΔABC**,** $BC = CF$**,**

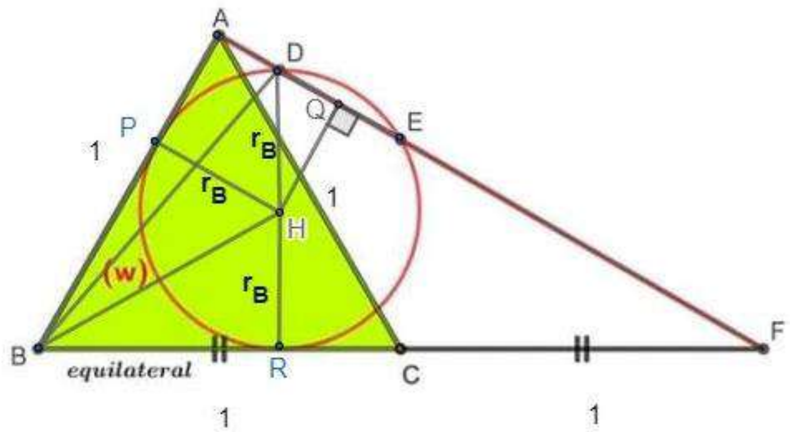
$AF \cap (w) = \{D, E\}$**.Prove:**

$$\frac{AD}{3^0} = \frac{AE}{3^1} = \frac{AF}{3^2}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Rajarshi Chakraborty-India



Let the sides of ΔABC be 1 unit length each.

$$\angle BAF = 90°$$

$PDER$ is the B – mixtilinear incircle of ΔABC

$$\text{Radius } r_B \text{ of } PDER = \frac{\Delta}{s^2}\cdot\frac{abc}{b(s-b)} = \frac{\sqrt{3}}{4}\cdot\frac{1}{\frac{9}{4}\cdot\frac{1}{2}} = \frac{2}{3\sqrt{3}}$$

$$PB = \frac{ac}{s} = \frac{2}{3}, \qquad AP = \frac{1}{3}$$

$$DQ = \sqrt{r_b^2 - AP^2} = \sqrt{\left(\frac{2}{3\sqrt{3}}\right)^2 - \left(\frac{1}{3}\right)^2} = \frac{1}{3\sqrt{3}}$$

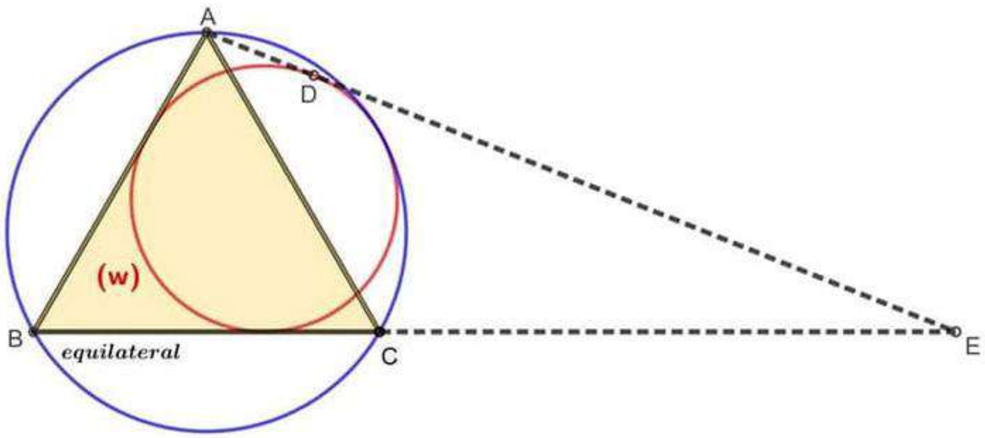
$$DE = 2\cdot DQ = \frac{2}{3\sqrt{3}}, \qquad AD = r_B - DQ = \frac{2}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{1}{3\sqrt{3}}$$

$$AE = AD + DE = \frac{2}{3\sqrt{3}} + \frac{1}{3\sqrt{3}} = \frac{3}{3\sqrt{3}}$$

$$AF = \sqrt{BF^2 - AB^2} = \sqrt{2^2 - 1} = \sqrt{3} = \frac{9}{3\sqrt{3}}. \text{ Thus } \frac{AD}{3^0} = \frac{AE}{3^1} = \frac{AF}{3^2}$$

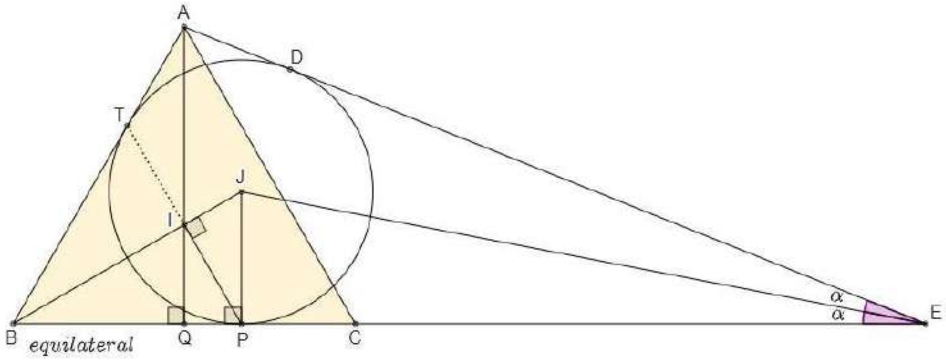
1353. (w) − mixtilinear incircle of ΔABC, D touch point.Prove:

$$\frac{AD}{AE} = \frac{1}{7}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Rodrigo Santos-Brazil



$$AB = AC = CA = x$$

D, T, P: touch points

$$BI = \frac{2}{3} \cdot AQ = \frac{2}{3} \cdot \frac{x\sqrt{3}}{2} = \frac{x\sqrt{3}}{3} \Rightarrow BP = BI \cdot \sec 30^\circ = \frac{2x}{3} \Rightarrow$$

$$\Rightarrow \begin{cases} PC = x - \frac{2x}{3} = \frac{x}{3} = AT = AD \\ QP = \frac{2x}{3} - \frac{x}{2} = \frac{x}{6} \end{cases}, \Delta BPJ: PJ = BP \cdot \tan 30^\circ = \frac{2x\sqrt{3}}{9}$$

$$\Delta AQE\ (ED = EP = t): \cos 2\alpha = \frac{t + \frac{x}{t}}{t + \frac{x}{3}} = \frac{6t + x}{6t + 2x}$$

$$\sin 2\alpha = \frac{\frac{x\sqrt{3}}{2}}{t + \frac{x}{3}} = \frac{3x\sqrt{3}}{6t + 2x},\ \Delta EJP:\ \tan \alpha = \frac{\frac{2x\sqrt{3}}{9}}{t} \Rightarrow \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \frac{2x\sqrt{3}}{9t}$$

$$\frac{\frac{3x\sqrt{3}}{6t + 2x}}{1 + \frac{6t + x}{6t + 2x}} = \frac{2x\sqrt{3}}{9t},\qquad \frac{3}{12t + 3x} = \frac{2}{9t} \therefore t = 2x = DE$$

So,

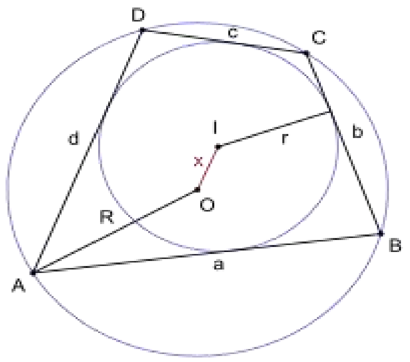
$$\frac{AD}{AE} = \frac{\frac{x}{3}}{\frac{x}{3} + 2x} \therefore \frac{AD}{AE} = \frac{1}{7}$$

1354. If a, b, c, d are sides in a bicentric quadrilateral then :

$$abc + abd + acd + bcd < 2\sqrt{\pi}(2 + \pi)R^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco.



We have :

$$a = 2R\sin\frac{\angle AOB}{2},\ \ b = 2R\sin\frac{\angle BOC}{2},\ \ c = 2R\sin\frac{\angle COD}{2},\ \ d = 2R\sin\frac{\angle DOA}{2}.$$

Then : $a + b + c + d = 2R\left(\sin\frac{\angle AOB}{2} + \sin\frac{\angle BOC}{2} + \sin\frac{\angle COD}{2} + \sin\frac{\angle DOA}{2}\right)$

$$\overset{Jensen}{\leq} 2R.4\sin\left(\frac{\angle AOB + \angle BOC + \angle COD + \angle DOA}{8}\right) = 8R.\sin\left(\frac{\pi}{4}\right) = 4\sqrt{2}R\ \ (1)$$

Now, using Maclaurin's inequality and the result (1), we get :

$$abc+abd+acd+bcd \le 4\left(\frac{a+b+c+d}{4}\right)^3 \le 4\left(\sqrt{2}R\right)^3 = 8\sqrt{2}R^3$$

$$= 2\sqrt{2}.4R^3 < 2\sqrt{\pi}(2+\pi)R^3$$

Solution 2 by Adrian Popa-Romania

$$\left.\begin{array}{r} ABCD - Tangential \Rightarrow a+c=b+d \\ a+b+c+d=2p \end{array}\right\} \Rightarrow a+c=b+d=p$$

$$abc+abd+acd+bcd = ac(b+d)+bd(a+c) =$$

$$= acp+bdp = p(ac+bd) \overset{MG-MA}{\le} p\left[\left(\frac{a+c}{2}\right)^2+\left(\frac{b+d}{2}\right)^2\right] =$$

$$= p\left(\frac{p^2}{4}+\frac{p^2}{4}\right) = p\cdot\frac{p^2}{2} = \frac{p^3}{2} \quad (1)$$

In a bicentric quadrilateral we have:

$$\left.\begin{array}{r} Blundon-Eddy\ inequality: p \le \sqrt{4R^2+r^2}+r \\ R \ge r\sqrt{2} \Rightarrow r \le \dfrac{R}{\sqrt{2}} \end{array}\right\} \Rightarrow$$

$$\Rightarrow p \le \sqrt{4R^2+\frac{R^2}{2}}+\frac{R}{\sqrt{2}} = \sqrt{\frac{9R^2}{2}}+\frac{R}{\sqrt{2}} = \frac{3R}{\sqrt{2}}+\frac{R}{\sqrt{2}} = \frac{4R}{\sqrt{2}} = \frac{4R\sqrt{2}}{2}$$

$$= 2R\sqrt{2} \quad (2)$$

From (1) and (2) $\Rightarrow abc+abd+acd+bcd \le \frac{(2R\sqrt{2})^3}{2} = \frac{8R^3\cdot 2\sqrt{2}}{2} =$

$$\left.\begin{array}{r} = 8R^3\sqrt{2} \overset{?}{<} 2\sqrt{\pi}(2+\pi)R^3 \\ 2\sqrt{\pi}(2+\pi) \overset{MA\ge MG}{>} 2\sqrt{\pi}\cdot 2\sqrt{2\pi} = 4\pi\sqrt{2} > 4\cdot 2\sqrt{2} = 8\sqrt{2} \end{array}\right\} \Rightarrow$$

$\Rightarrow 8\sqrt{2}R^2 < 2\sqrt{\pi}(2+\pi)R^3 \Rightarrow$ the inequality from enunciation holds.

1355. In $\Delta ABC, I -$ incenter, $N_a -$ Nagel's point.

If B, I, N_a, C are concyclic then prove that

$$2R - r \ge h_a \ge 3r$$

Proposed by Thanasis Gakopoulos-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $BC = a, CA = b, AB = c$ and
$2s = a + b + c$. In barycentric coordinates relative to (A, B, C),
we have $I = (a : b : c)$ and $N_a = (s - a : s - b : s - c)$.
We know that the circle Γ through B, I, C has an equation of the form
$$a^2yz + b^2zx + c^2xy = (x + y + z)(\alpha x + \beta y + \gamma z).$$
Expressing that $B(0 : 1 : 0), C(0 : 0 : 1), I(a : b : c)$ are on this circle, we obtain that the equation of Γ is
$$a^2yz + b^2zx + c^2xy = (x + y + z)xbc.$$
$$N_a \in \Gamma \Rightarrow a^2(s - b)(s - c) + b^2(s - b)(s - c) + c^2(s - b)(s - c) = s(s - a)bc$$
$$\Leftrightarrow s^2(a^2 + b^2 + c^2) - s[(a + b + c)(ab + bc + ca) - 3abc] + abc(a + b + c) = s^2bc - sabc$$
$$\Leftrightarrow s^2.2(s^2 - r^2 - 4Rr) - s[2s(s^2 + r^2 + 4Rr) - 3.4Rsr] + 4Rsr.2s = s^2bc - s.4Rsr$$
$$\Leftrightarrow bc = 4r(2R - r) \Leftrightarrow h_a = \frac{bc}{2R} = \frac{2r(2R - r)}{R}.$$
By Euler's inequality, $R \ge 2r$, we obtain
$$2R - r \ge \frac{2r(2R - r)}{R} = h_a = 3r + \frac{r(R - 2r)}{R} \ge 3r.$$
So the proof is complete. Equality holds iff ΔABC is equilateral.

1356. In ΔABC the following relationship holds:

$$\frac{1}{3}\left(\frac{m_a}{h_c} + \frac{w_a}{h_b} + \frac{r_a}{h_a}\right) \ge \sqrt[3]{\frac{R}{2r}} + \frac{1}{3}max\left\{\left(\sqrt{\frac{m_a}{h_c}} - \sqrt{\frac{w_a}{h_b}}\right)^2, \left(\sqrt{\frac{w_a}{h_b}} - \sqrt{\frac{r_a}{h_a}}\right)^2, \left(\sqrt{\frac{r_a}{h_a}} - \sqrt{\frac{m_a}{h_c}}\right)^2\right\}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have, for all $x, y, z > 0$

$$\frac{1}{3}(x + y + z) = \frac{1}{3}(z + \sqrt{xy} + \sqrt{xy}) + \frac{1}{3}(\sqrt{x} - \sqrt{y})^2 \ge \sqrt[3]{xyz} + \frac{1}{3}(\sqrt{x} - \sqrt{y})^2 \quad \text{(and analogs).}$$

Then

$$\frac{1}{3}(x + y + z) \ge \sqrt[3]{xyz} + \frac{1}{3}max\left\{(\sqrt{x} - \sqrt{y})^2, (\sqrt{y} - \sqrt{z})^2, (\sqrt{z} - \sqrt{x})^2\right\}.$$

Taking $x = \frac{m_a}{h_c}, y = \frac{w_a}{h_b}, z = \frac{r_a}{h_a}$, we have

$$\frac{1}{3}\left(\frac{m_a}{h_c}+\frac{w_a}{h_b}+\frac{r_a}{h_a}\right)\geq\sqrt[3]{\frac{m_aw_ar_a}{h_ah_bh_c}}+\frac{1}{3}max\left\{\left(\sqrt{\frac{m_a}{h_c}}-\sqrt{\frac{w_a}{h_b}}\right)^2,\left(\sqrt{\frac{w_a}{h_b}}-\sqrt{\frac{r_a}{h_a}}\right)^2,\left(\sqrt{\frac{r_a}{h_a}}-\sqrt{\frac{m_a}{h_c}}\right)^2\right\}$$

Also, we have

$$h_ah_bh_c=\frac{2s^2r^2}{R}\text{ and }m_aw_ar_a\geq\frac{b+c}{2}\cos\frac{A}{2}.\frac{2bc}{b+c}\cos\frac{A}{2}.r_a=bc.\frac{s(s-a)}{bc}.\frac{sr}{s-a}=s^2r.$$

Then

$$\frac{m_aw_ar_a}{h_ah_bh_c}\geq\frac{R}{2r}.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.

1357. In any Δ ABC, the following relationship holds :

$$\frac{r_a}{4r_b^2+r_br_c+4r_c^2}+\frac{r_b}{4r_c^2+r_cr_a+4r_a^2}+\frac{r_c}{4r_a^2+r_ar_b+4r_b^2}\geq\frac{2}{9R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{4r_b^2+r_br_c+4r_c^2}+\frac{r_b}{4r_c^2+r_cr_a+4r_a^2}+\frac{r_c}{4r_a^2+r_ar_b+4r_b^2}$$

$$=\sum_{cyc}\frac{r_a^4}{4r_a^3r_b^2+r_a^3r_br_c+4r_a^3r_c^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}r_a^2\right)^2}{4\sum_{cyc}x^3y^2+4\sum_{cyc}x^2y^3+xyz\sum_{cyc}x^2}$$

$$(x=r_a,y=r_b,z=r_c)\rightarrow(1)$$

$$\text{Now, }4\sum_{cyc}x^3y^2+4\sum_{cyc}x^2y^3+xyz\sum_{cyc}x^2$$

$$=4\sum_{cyc}\left(x^2y^2\left(\sum_{cyc}x-z\right)\right)+xyz\sum_{cyc}x^2$$

$$=4\left(\sum_{cyc}x\right)\left(\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)\right)-4xyz\left(\sum_{cyc}xy\right)+xyz\sum_{cyc}x^2$$

$$=4\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)^2-8xyz\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)-4xyz\left(\sum_{cyc}xy\right)+xyz\sum_{cyc}x^2$$

$$= 4(4R+r)s^4 - xyz\left(\left(7\sum_{cyc} x^2 + 14\sum_{cyc} xy\right) + 6\sum_{cyc} xy\right)$$

$$= 4(4R+r)s^4 - 7rs^2(4R+r)^2 - 6rs^4 \therefore (1) \Rightarrow \text{LHS} \geq$$

$$\frac{\left((4R+r)^2 - 2s^2\right)^2}{4(4R+r)s^4 - 7rs^2(4R+r)^2 - 6rs^4} \overset{?}{\geq} \frac{2}{9R}$$

$$\Leftrightarrow (4R+4r)s^4 + 2304R^5 + 2304R^4r + 864R^3r^2 + 144R^2r^3 + 9Rr^4$$

$$\underset{(*)}{\overset{?}{\geq}} (576R^3 + 64R^2r - 76Rr^2 - 14r^3)s^2 \text{ and } \because (4R+4r)\left(s^2 - 16Rr + 5r^2\right)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \geq (4R+4r)\left(s^2 - 16Rr + 5r^2\right)^2$$

$$\Leftrightarrow 2304R^5 + 2304R^4r - 160R^3r^2 - 240R^2r^3 + 549Rr^4 - 100r^5$$

$$\overset{(**)}{\geq} (576R^3 - 64R^2r - 164Rr^2 + 26r^3)s^2$$

$$\text{Again, RHS of } (**) \overset{\text{Rouche}}{\leq}$$

$$(576R^3 - 64R^2r - 164Rr^2 + 26r^3)\begin{pmatrix} 2R^2 + 10Rr - r^2 + \\ 2(R-2r) * \sqrt{R^2 - 2Rr} \end{pmatrix} \overset{?}{\leq} \text{LHS of } (**)$$

$$\Leftrightarrow 1152R^5 - 3328R^4r + 1384R^3r^2 + 1284R^2r^3 + 125Rr^4 - 74r^5 \overset{?}{\geq}$$
$$2(R-2r)(576R^3 - 64R^2r - 164Rr^2 + 26r^3) * \sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow (R-2r)(1152R^4 - 1024R^3r - 664R^2r^2 - 44Rr^3 + 37r^4) \overset{?}{\geq}$$
$$2(R-2r)(576R^3 - 64R^2r - 164Rr^2 + 26r^3) * \sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow 1152R^4 - 1024R^3r - 664R^2r^2 - 44Rr^3 + 37r^4 \overset{?}{>}$$
$$2(576R^3 - 64R^2r - 164Rr^2 + 26r^3) * \sqrt{R^2 - 2Rr}$$

$$\begin{pmatrix} \because R - 2r \overset{\text{Euler}}{\geq} 0 \text{ and } 1152R^4 - 1024R^3r - 664R^2r^2 - 44Rr^3 + 37r^4 \\ = (R-2r)(1152R^3 + 1280R^2r + 1896Rr^2 + 3748r^3) + 7533r^4 \overset{\text{Euler}}{\geq} 7533r^4 > 0 \end{pmatrix}$$

$$\Leftrightarrow (1152R^4 - 1024R^3r - 664R^2r^2 - 44Rr^3 + 37r^4)^2 \overset{?}{>}$$
$$4(R^2 - 2Rr)(576R^3 - 64R^2r - 164Rr^2 + 26r^3)^2$$

$$\Leftrightarrow 589824t^7 - 331776t^6 - 423936t^5 + 929536t^4 + 205312t^3 - 118128t^2$$
$$+2152t + 1369 \overset{?}{>} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\begin{pmatrix} 589824t^6 + 847872t^5 + 1271808t^4 + 3473152t^3 + \\ 7151616t^2 + 14185104t + 28372360 \end{pmatrix}$$
$$+56746089 \overset{?}{>} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{r_a}{4r_b^2 + r_br_c + 4r_c^2} + \frac{r_b}{4r_c^2 + r_cr_a + 4r_a^2} + \frac{r_c}{4r_a^2 + r_ar_b + 4r_b^2} \geq \frac{2}{9R}$$

$$\forall \Delta ABC,'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1358. In any Δ ABC, the following relationship holds :

$$\frac{m_a}{m_b(m_a^3+m_b^3)}+\frac{m_b}{m_c(m_b^3+m_c^3)}+\frac{m_c}{m_a(m_c^3+m_a^3)}\ge\frac{16r^2}{9(81R^5-2560r^5)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2\overset{?}{\le}\frac{b^3+c^3+abc}{4a}\Leftrightarrow a(2b^2+2c^2-a^2)\overset{?}{\le}b^3+c^3+abc$$

$$\Leftrightarrow\sum_{cyc}a^3+abc\overset{?}{\ge}2a(b^2+c^2)$$

$$\Leftrightarrow\sum_{cyc}(y+z)^3+\prod_{cyc}(y+z)\overset{?}{\ge}2(y+z)((z+x)^2+(x+y)^2)$$

$$\left(\begin{matrix}x=s-a,y=s-b,z=s-c\Rightarrow x+y+z=3s-2s=s\\ \Rightarrow a=y+z,b=z+x,c=x+y;x,y,z>0\end{matrix}\right)$$

$$\Leftrightarrow x^3+y^2z+yz^2\overset{?}{\ge}3xyz\to\text{true via A}-\text{G}\therefore m_a^2\le\frac{b^3+c^3+abc}{4a}$$

$$\Rightarrow m_a^3\overset{Panaitopol}{\le}\frac{b^3+c^3+abc}{4a}.\frac{Rs}{a}=\frac{Rs}{4.16R^2r^2s^2}.b^2c^2(b^3+c^3+abc)$$

$$\Rightarrow m_a^3\le\frac{1}{64Rr^2s}\left(b^2c^2\left(\sum_{cyc}a^3+abc\right)-a^3b^2c^2\right)\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}m_a^3\le\frac{1}{64Rr^2s}\left(\left(\sum_{cyc}a^3+abc\right)\left(\sum_{cyc}b^2c^2\right)-16R^2r^2s^2(2s)\right)$$

$$\overset{Goldstone}{\le}\frac{(2s(s^2-6Rr-3r^2)+4Rrs)(4R^2s^2)-16R^2r^2s^2(2s)}{64Rr^2s}$$

$$=\frac{2s.4R^2s^2(s^2-4Rr-7r^2)}{64Rr^2s}\Rightarrow\sum_{cyc}m_a^3\le\frac{Rs^2(s^2-4Rr-7r^2)}{8r^2}$$

$$\therefore\frac{m_a}{m_b(m_a^3+m_b^3)}+\frac{m_b}{m_c(m_b^3+m_c^3)}+\frac{m_c}{m_a(m_c^3+m_a^3)}=\sum_{cyc}\frac{\sqrt{\frac{m_a}{m_b}}^2}{m_a^3+m_b^3}\overset{Bergstrom}{\ge}$$

$$\frac{\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right)^2}{2\sum_{cyc}m_a^3}\overset{A-G}{\ge}\frac{9}{2\sum_{cyc}m_a^3}\ge\frac{9*16r^2}{R(4s^2)(s^2-4Rr-7r^2)}\overset{\substack{Mitrinovic\\ and\\ Gerretsen}}{\ge}$$

$$\frac{9*16r^2}{27R^3(4R^2-4r^2)}\overset{?}{\ge}\frac{16r^2}{9(81R^5-2560r^5)}\Leftrightarrow239t^5+4t^3-7680\overset{?}{\ge}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(239t^4+478t^3+960t^2+1920t+3840)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2$$

$$\Rightarrow \frac{m_a}{m_b(m_a^3+m_b^3)}+\frac{m_b}{m_c(m_b^3+m_c^3)}+\frac{m_c}{m_a(m_c^3+m_a^3)}\geq\frac{16r^2}{9(81R^5-2560r^5)}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1359. In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{r_a\left(\frac{2(bm_c+cm_b-2F)}{a}-r_a\right)}\leq\sum_{cyc}n_a$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We prove the result by using the following inequality

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}\leq\frac{R}{r}\quad\text{(and analogs)}$$

(see, Bogdan Fuştei, Mohamed Amine Ben Ajiba, " *New Triangle Inequalities With Brocard's Angle* ", Lemma 12, 6 − 7, www.ssmrmh.ro).

$$r_a\left(\frac{2(bm_c+cm_b-2F)}{a}-r_a\right)=2r_ah_a\left(\frac{m_c}{h_b}+\frac{m_b}{h_c}-1\right)-r_a{}^2$$

$$\leq 2s\tan\frac{A}{2}\cdot\frac{2sr}{2R\sin A}\cdot\frac{R}{r}-r_a{}^2-2r_ah_a=s^2\sec^2\frac{A}{2}-s^2\tan^2\frac{A}{2}-2r_ah_a=s^2-\frac{4s^2r^2}{a(s-a)}$$

$$=s^2-\frac{4s(s-b)(s-c)}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s(s-a)+\frac{s(b-c)^2}{a}=n_a{}^2.$$

$$\Rightarrow\sqrt{r_a\left(\frac{2(bm_c+cm_b-2F)}{a}-r_a\right)}\leq n_a\quad\text{(and analogs)}$$

Therefore

$$\sum_{cyc}\sqrt{r_a\left(\frac{2(bm_c+cm_b-2F)}{a}-r_a\right)}\leq\sum_{cyc}n_a.$$

Equality holds iff ΔABC is equilateral.

1360. In any Δ ABC, the following relationship holds :

$$\frac{(h_a+w_b+m_c)^5}{r_a^5}+\frac{(h_b+w_c+m_a)^5}{r_b^5}+\frac{(h_c+w_a+m_b)^5}{r_b^5}\geq\frac{3\cdot 6^5\cdot r^5}{81R^5-2560r^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} r_a^5=\left(\sum_{cyc} r_a\right)^5-5\left(\sum_{cyc} r_a^2+\sum_{cyc} r_a r_b\right)\left(\prod_{cyc}(r_b+r_c)\right)$$

$$\overset{\substack{\text{Leuenberger + Euler,}\\ \text{Cesaro,}\\ \text{A-G}}}{\leq}\left(4R+\frac{R}{2}\right)^5-5\left(2\sum_{cyc} r_a r_b\right)(8r_a r_b r_c)=\left(\frac{9R}{2}\right)^5-5.2s^2.8rs^2$$

$$\overset{\text{Mitrinovic}}{\leq}\left(\frac{9R}{2}\right)^5-5.2.8r.729r^4=\frac{729(81R^5-2560r^5)}{32}$$

$$\therefore\sum_{cyc} r_a^5\overset{(\bullet)}{\leq}\frac{729(81R^5-2560r^5)}{32}$$

$$\text{Now, }\frac{(h_a+w_b+m_c)^5}{r_a^5}+\frac{(h_b+w_c+m_a)^5}{r_b^5}$$

$$+\frac{(h_c+w_a+m_b)^5}{r_b^5}\overset{\text{Holder}}{\geq}\frac{(\sum_{cyc} h_a+\sum_{cyc} w_a+\sum_{cyc} m_a)^5}{27\sum_{cyc} r_a^5}\overset{\text{via }(\bullet)}{\geq}\frac{32\left(3*\sum_{cyc}\frac{2rs}{a}\right)^5}{3^9*(81R^5-2560r^5)}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{32*3^5*\left(\frac{2rs*9}{2s}\right)^5}{3^9*(81R^5-2560r^5)}=\frac{2^5*3^6*r^5}{81R^5-2560r^5}=\frac{3*6^5*r^5}{81R^5-2560r^5}$$

$\forall\,\Delta$ ABC," =" iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $w_a, m_a \geq h_a$ (and analogs) and

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r},\text{ then we have}$$

$$\sum_{cyc}\frac{(h_a+w_b+m_c)^5}{r_a{}^5}\geq\sum_{cyc}\frac{(h_a+h_b+h_c)^5}{r_a{}^5}\overset{Holder}{\geq}\frac{(h_a+h_b+h_c)^5}{3^4}.\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)^5$$

$$=\frac{1}{3^4}.\left((h_a+h_b+h_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\right)^5\overset{CBS}{\geq}\frac{1}{3^4}.9^5\overset{?}{\geq}\frac{3.6^5.r^5}{81R^5-2560r^5}\Leftrightarrow R^5\geq 32r^5,$$

which is true by Euler's inequality $R\geq 2r$. Equality holds iff ΔABC is equilateral.

1361. In any Δ ABC, the following relationship holds :

$$(4R+r)^2\cdot\frac{R+2r}{R+r}\le\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}\le\frac{4}{3}(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}=\sum_{cyc}\frac{r^2s^2.bc}{(s-a)^2(s-b)(s-b)}=\frac{r^2s^3}{r^2s}\sum_{cyc}\frac{bc}{s(s-a)}=s^2\sum_{cyc}\sec^2\frac{A}{2}$$

$$=s^2.\frac{s^2+(4R+r)^2}{s^2}\ \therefore\ \sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}\overset{(*)}{=}s^2+(4R+r)^2$$

$$\therefore\ (4R+r)^2.\frac{R+2r}{R+r}\le\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}\overset{\text{via }(*)}{\Leftrightarrow}s^2+(4R+r)^2\ge(4R+r)^2.\frac{R+2r}{R+r}$$

$$\Leftrightarrow (R+r)s^2\overset{(i)}{\ge}r(4R+r)^2$$

$$\text{Now, }(R+r)s^2\overset{\text{Gerretsen}}{\ge}(R+r)\left(16Rr-5r^2\right)\overset{?}{\ge}r(4R+r)^2\Leftrightarrow 3r(R-2r)\overset{?}{\ge}0$$

$$\to\text{ true via Euler}\Rightarrow(*)\text{ is true}\ \therefore\ (4R+r)^2.\frac{R+2r}{R+r}\le\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}$$

$$\text{Again, via }(*),\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}=s^2+(4R+r)^2\overset{\text{Doucet / Trucht}}{\le}\frac{1}{3}(4R+r)^2+(4R+r)^2$$

$$=\frac{4}{3}(4R+r)^2\ \therefore\ \sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}\le\frac{4}{3}(4R+r)^2$$

$$\therefore\ (4R+r)^2.\frac{R+2r}{R+r}\le\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}\le\frac{4}{3}(4R+r)^2$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)

1362. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{\lambda(4R+r)}{3r}\ge\frac{3}{2}(2\lambda+1)\ \forall\ \lambda\ge\frac{3}{4}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}=\sum_{cyc}\frac{r_ar_b}{r_a^2+r_b^2}=rs^2\sum_{cyc}\frac{1}{r_c(r_a^2+r_b^2)}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{9rs^2}{\sum_{cyc}\left(r_ar_b(\sum_{cyc}r_a-r_c)\right)}=\frac{9rs^2}{(\sum_{cyc}r_a)(\sum_{cyc}r_ar_b)-3r_ar_br_c}=\frac{9rs^2}{(4R+r)s^2-3rs^2}$$

$$\Rightarrow\sum_{cyc}\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{\lambda(4R+r)}{3r}\geq\frac{9r}{4R-2r}+\frac{\lambda(4R+r)}{3r}\overset{?}{\geq}\frac{3}{2}(2\lambda+1)$$

$$\Leftrightarrow\lambda\left(\frac{4R+r}{3r}-3\right)\overset{?}{\geq}\frac{3}{2}-\frac{9r}{4R-2r}\Leftrightarrow\lambda\left(\frac{4R-8r}{3r}\right)\underset{(*)}{\overset{?}{\geq}}\frac{3R-6r}{2R-r}$$

$$\because\lambda\geq\frac{3}{4}\text{ and }\frac{4R-8r}{3r}\overset{\text{Euler}}{\geq}0\therefore\lambda\left(\frac{4R-8r}{3r}\right)\geq\frac{R-2r}{r}\overset{?}{\geq}\frac{3R-6r}{2R-r}$$

$$\Leftrightarrow 2R-r\overset{?}{\geq}3r\left(\because R-2r\overset{\text{Euler}}{\geq}0\right)\Leftrightarrow 2R\overset{?}{\geq}4r\to\text{true via Euler}\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{\lambda(4R+r)}{3r}\geq\frac{3}{2}(2\lambda+1)\ \forall\lambda\geq\frac{3}{4}$$

and $\forall\,\Delta\,ABC$,'' ='' iff $\Delta\,ABC$ is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}+\frac{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}{4\tan\frac{B}{2}\tan\frac{C}{2}}\geq 1\ \text{(and analogs)}$$

Then

$$\sum_{cyc}\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}+\frac{\lambda(4R+r)}{3r}\geq\sum_{cyc}\left(1-\frac{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}{4\tan\frac{B}{2}\tan\frac{C}{2}}\right)+\frac{\lambda(4R+r)}{3r}$$

$$=3-\left(\frac{\left(\sum_{cyc}\tan\frac{A}{2}\right)\left(\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}\right)}{4\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}-\frac{3}{4}\right)+\frac{\lambda(4R+r)}{3r}$$

$$=\frac{15}{4}-\frac{4R+r}{4r}+\frac{\lambda(4R+r)}{3r}=\frac{15}{4}+\frac{4R+r}{3r}\left(\lambda-\frac{3}{4}\right)\overset{\text{Euler}}{\geq}\frac{15}{4}+3\left(\lambda-\frac{3}{4}\right)=\frac{3}{2}(2\lambda+1),$$

as desired. Equality holds iff ΔABC is equilateral.

1363. In any $\Delta\,ABC$ with $\omega\to$

Brocard's angle, the following relationship holds :

$$\frac{1}{\sin\omega}\geq max\left(\sqrt{\frac{2R}{r}},\sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sin\omega}\geq\sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}}\Leftrightarrow\frac{\sum_{cyc}a^2b^2}{4r^2s^2}\geq\frac{2R}{r}\cdot\frac{rs^2(s^2+2Rr+r^2)}{16Rr^2s^2}$$
$$\Leftrightarrow 2\left((s^2+4Rr+r^2)^2-16Rrs^2\right)\geq s^2(s^2+2Rr+r^2)$$
$$\Leftrightarrow s^4-(18Rr-3r^2)s^2+2r^2(4R+r)^2\overset{(*)}{\geq}0$$
$$and\ \because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore \text{in order to prove }(*),\text{it suffices to prove}:$$
$$\text{LHS of }(*)\geq(s^2-16Rr+5r^2)^2\Leftrightarrow(14R-7r)s^2\overset{(**)}{\geq}r(224R^2-176Rr+23r^2)$$
$$and\ again,\text{LHS of }(**)\overset{\text{Rouche}}{\geq}(14R-7r)\left(\begin{array}{c}2R^2+10Rr-r^2\\-2(R-2r)*\sqrt{R^2-2Rr}\end{array}\right)$$
$$\overset{?}{\geq}r(224R^2-176Rr+23r^2)$$
$$\Leftrightarrow(R-2r)(14R^2-21Rr+4r^2)\overset{?}{\geq}(14R-7r)(R-2r)*\sqrt{R^2-2Rr}$$
$$\Leftrightarrow(14R^2-21Rr+4r^2)^2\overset{?}{\geq}(R^2-2Rr)(14R-7r)^2\ \left(\because R-2r\overset{\text{Euler}}{\geq}0\right)$$
$$\Leftrightarrow 2r^2(18R(R-2r)+38R^2+Rr+8r^2)\overset{?}{\geq}0\to\text{true}\Rightarrow(**)\Rightarrow(*)\text{ is true}$$
$$\therefore\boxed{\frac{1}{\sin\omega}\overset{(\bullet)}{\geq}\sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}}}\overset{w_a\leq\sqrt{s(s-a)}\text{ and analogs}}{\geq}\sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{rs^2}{\sqrt{s(s-a)}*\sqrt{s(s-b)}*\sqrt{s(s-c)}}}$$
$$=\sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{rs^2}{s.rs}}\therefore\frac{1}{\sin\omega}\overset{(\bullet\bullet)}{\geq}\sqrt{\frac{2R}{r}}\therefore(\bullet),(\bullet\bullet)\Rightarrow$$
$$\frac{1}{\sin\omega}\geq max\left(\sqrt{\frac{2R}{r}},\sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}}\right),''=''\ \text{iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $w_a=\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{r_b r_c}$ (and analogs) and

$$\sin\omega=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}},\text{then we have}$$

$$\frac{1}{\sin\omega} \ge \sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}} \Leftrightarrow \frac{a^2b^2+b^2c^2+c^2a^2}{4F^2} \ge \frac{2R}{r}\cdot\frac{(a+b)(b+c)(c+a)}{8abc}$$

$$\Leftrightarrow 4(a^2b^2+b^2c^2+c^2a^2) \ge s(a+b)(b+c)(c+a).$$

Using the following sustitution $a=y+z, b=z+x, c=x+z, x, y, z > 0$, **and after expanding and simplifying we get the equivalent expression**

$$2[x^4+y^4+z^4+xyz(x+y+z)] \ge xy(x+y)^2+yz(y+z)^2+zx(z+x)^2$$

By the fourth degree Schur's inequality, we have

$$2[x^4+y^4+z^4+xyz(x+y+z)] \ge xy.2(x^2+y^2)+yz.2(y^2+z^2)+zx.2(z^2+x^2)$$
$$\ge xy(x+y)^2+yz(y+z)^2+zx(z+x)^2,$$

Then

$$\frac{1}{\sin\omega} \ge \sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}}.$$

and since $w_a = \frac{2\sqrt{bc}}{b+c}.\sqrt{r_b r_c} \le \sqrt{r_b r_c}$ **(and analogs), then we have**

$$w_a w_b w_c \le r_a r_b r_c,\ \text{and}$$

$$\frac{1}{\sin\omega} \ge \sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}} = max\left\{\sqrt{\frac{2R}{r}}, \sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{r_a r_b r_c}{w_a w_b w_c}}\right\}.$$

Equality holds iff ΔABC **is equilateral.**

1364.

In any Δ ABC with ω → Brocard's angle, the following relationship holds :

$$\frac{2}{\sin\omega} \ge \sqrt{\frac{R(a+b)(b+c)(c+a)}{rabc}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{2}{\sin\omega} \ge \sqrt{\frac{R(a+b)(b+c)(c+a)}{rabc}} \Leftrightarrow \frac{4\sum_{cyc}a^2b^2}{4r^2s^2} \ge \frac{R*2s(s^2+2Rr+r^2)}{r*4Rrs}$$

$$\Leftrightarrow 2\sum_{cyc}a^2b^2 \ge s^2(s^2+2Rr+r^2)$$

$$\Leftrightarrow 2\left((s^2+4Rr+r^2)^2-16Rrs^2\right) \ge s^2(s^2+2Rr+r^2)$$

$$\Leftrightarrow s^4-(18Rr-3r^2)s^2+2r^2(4R+r)^2 \overset{(*)}{\ge} 0 \text{ and } \because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0$$

$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*) \geq \left(s^2 - 16Rr + 5r^2\right)^2$$

$$\Leftrightarrow (14R - 7r)s^2 \overset{(**)}{\geq} r\left(224R^2 - 176Rr + 23r^2\right) \text{ and again, LHS of } (**) \overset{\text{Rouche}}{\geq}$$

$$(14R - 7r)\left(2R^2 + 10Rr - r^2 - 2(R - 2r) * \sqrt{R^2 - 2Rr}\right)$$

$$\overset{?}{\geq} r\left(224R^2 - 176Rr + 23r^2\right)$$

$$\Leftrightarrow (R - 2r)\left(14R^2 - 21Rr + 4r^2\right) \overset{?}{\geq} (14R - 7r)(R - 2r) * \sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow \left(14R^2 - 21Rr + 4r^2\right)^2 \overset{?}{\geq} \left(R^2 - 2Rr\right)(14R - 7r)^2 \left(\because R - 2r \overset{\text{Euler}}{\geq} 0\right)$$

$$\Leftrightarrow 2r^2\left(18R(R - 2r) + 38R^2 + Rr + 8r^2\right) \overset{?}{\geq} 0 \to \text{true} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{2}{\sin\omega} \geq \sqrt{\frac{R(a+b)(b+c)(c+a)}{rabc}} \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $\sin\omega = \dfrac{2sr}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$, **then the desired inequality is equivalent to**

$$4(a^2b^2 + b^2c^2 + c^2a^2) \geq s(a+b)(b+c)(c+a).$$

Using the following sustitution

$$a = y + z, b = z + x, c = x + z, x, y, z > 0, \text{and after expanding}$$

and simplifying we get the equivalent expression

$$2[x^4 + y^4 + z^4 + xyz(x+y+z)] \geq xy(x+y)^2 + yz(y+z)^2 + zx(z+x)^2$$

By the fourth degree Schur's inequality, we have

$$2[x^4 + y^4 + z^4 + xyz(x+y+z)] \geq xy.2(x^2 + y^2) + yz.2(y^2 + z^2) + zx.2(z^2 + x^2)$$

$$\geq xy(x+y)^2 + yz(y+z)^2 + zx(z+x)^2,$$

So the proof is complete. Equality holds iff ΔABC **is equilateral.**

1365. In ΔABC the following relationship holds:

$$\frac{2s+a}{4s-a} + \frac{2s+b}{4s-b} + \frac{2s+c}{4s-c} \geq \frac{48}{5} \cdot \left(\frac{r}{R}\right)^2$$

Proposed by Emil Popa-Romania

Solution by Daniel Sitaru-Romania

$$\frac{2s+a}{4s-a} + \frac{2s+b}{4s-b} + \frac{2s+c}{4s-c} =$$

$$= \sum_{cyc} \frac{a+b+c+a}{2a+2b+2c+a} = \sum_{cyc} \frac{2a+b+c}{3a+2b+2c} =$$

$$=\sum_{cyc}\frac{(2a+b+c)^2}{(3a+2b+2c)(2a+b+c)}=$$

$$=\sum_{cyc}\frac{(2a+b+c)^2}{2a^2+2b^2+2c^2+5ab+5ac+4bc}\overset{BERGSTROM}{\geq}\frac{(4a+4b+4c)^2}{6\sum_{cyc}a^2+14\sum_{cyc}ab}=$$

$$=\frac{16(a+b+c)^2}{6\cdot 2(s^2-r^2-4Rr)+14(s^2+r^2+4Rr)}=$$

$$=\frac{8\cdot(2s)^2}{13s^2+r^2+4Rr}\overset{GERRETSEN}{\geq}\frac{8\cdot 4s^2}{13(4R^2+4Rr+3r^2)+r^2+4Rr}\geq$$

$$\overset{MITRINOVIC}{\geq}\frac{32\cdot 27r^2}{52R^2+56Rr+40r^2}\overset{EULER}{\geq}$$

$$\geq\frac{32\cdot 27r^2}{52R^2+56R\cdot\frac{R}{2}+40\cdot\frac{R^2}{4}}=\frac{32\cdot 27r^2}{52R^2+28R^2+10R^2}=\frac{32\cdot 27r^2}{90R^2}=\frac{48}{5}\cdot\left(\frac{r}{R}\right)^2$$

Equality holds for $a=b=c$.

1366. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}\geq\frac{2r}{R}\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Notice that if $a\leq b\leq c$ then $h_a\geq h_b\geq h_c$ and $\csc\frac{A}{2}\geq\csc\frac{B}{2}\geq\csc\frac{C}{2}$.

Chebyshev's inequality affirms that

$$\sum_{cyc}\frac{{h_a}^2}{\sin^2\frac{A}{2}}\geq\frac{1}{3}\sum_{cyc}{h_a}^2\cdot\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}$$

$$\geq\frac{1}{3}\sum_{cyc}h_bh_c\cdot\frac{s^2+r^2-8Rr}{r^2}\overset{Gerretsen}{\geq}\frac{1}{3}\cdot\frac{2s^2r}{R}\cdot\frac{8R-4r}{r}\overset{Euler}{\geq}4s^2.$$

Also, we have

$$\frac{2r}{R}\cdot\sum_{cyc}\frac{{r_a}^2}{\sin^2\frac{A}{2}}=\frac{2r}{R}\cdot\sum_{cyc}\frac{F^2bc}{(s-a)^2(s-b)(s-c)}=\frac{2s^2r^3}{R}\cdot\sum_{cyc}\frac{bc(s-b)(s-c)}{(sr^2)^2}$$

$$\overset{AM-GM}{\leq}\frac{2}{Rr}\cdot\sum_{cyc}\frac{bc[(s-b)+(s-c)]^2}{4}=\frac{1}{2Rr}\cdot\sum_{cyc}bca^2=4s^2.$$

Therefore

$$\sum_{cyc}\frac{{h_a}^2}{\sin^2\frac{A}{2}}\geq 4s^2\geq\frac{2r}{R}\cdot\sum_{cyc}\frac{{r_a}^2}{\sin^2\frac{A}{2}}.$$

Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}=\sum_{cyc}\left(h_a^2\left(1+\cot^2\frac{A}{2}\right)\right)=\sum_{cyc}h_a^2+s^2\sum_{cyc}\frac{h_a^2}{r_a^2}$$

$$=\sum_{cyc}\frac{a^2b^2}{4R^2}+\frac{s^2}{r^2s^4}\sum_{cyc}h_a^2r_b^2r_c^2=\frac{1}{4R^2}\sum_{cyc}a^2b^2+\frac{4r^2s^4}{r^2s^2}\sum_{cyc}\frac{(s-a)^2}{a^2}$$

$$=\frac{1}{4R^2}\sum_{cyc}a^2b^2+4s^2\left(s^2\sum_{cyc}\frac{1}{a^2}-2s\sum_{cyc}\frac{1}{a}+3\right)$$

$$=\frac{1}{4R^2}\sum_{cyc}a^2b^2+\frac{4s^4}{16R^2r^2s^2}\sum_{cyc}a^2b^2-\frac{8s^3}{4Rrs}\sum_{cyc}ab+12s^2$$

$$=\frac{(s^2+r^2)\left((s^2+4Rr+r^2)-16Rrs^2\right)-8Rrs^2(s^2+4Rr+r^2)+48R^2r^2s^2}{4R^2r^2}$$

$$=\frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{4R^2r^2}\overset{(*)}{=}\sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}$$

$$\text{Again,}\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}=\sum_{cyc}\frac{r^2s^2.bc}{(s-a)^2(s-b)(s-b)}=\frac{r^2s^3}{r^2s}\sum_{cyc}\frac{bc}{s(s-a)}=s^2\sum_{cyc}\sec^2\frac{A}{2}$$

$$=s^2.\frac{s^2+(4R+r)^2}{s^2}\therefore\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}\overset{(**)}{=}s^2+(4R+r)^2$$

$$\therefore\text{ via }(*),(**)\sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}\geq\frac{2r}{R}\sum_{cyc}\frac{r_a^2}{\sin^2\frac{A}{2}}$$

$$\Leftrightarrow\frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{4R^2r^2}$$

$$\geq\frac{2r}{R}\left(s^2+(4R+r)^2\right)\Leftrightarrow s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-16Rr+3r^2)$$

$$-r^3(128R^3+48R^2r-r^3)\overset{(i)}{\geq}0$$

$$\text{Now, LHS of (i)}\overset{\text{Gerretsen}}{\geq}(16Rr-5r^2)s^4-(16Rr-3r^2)s^4$$

$$+r^2s^2(32R^2-16Rr+3r^2)-r^3(128R^3+48R^2r-r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow(32R^2-16Rr+3r^2)s^2-r(128R^3+48R^2r-r^3)\underset{(ii)}{\overset{?}{\geq}}2s^4$$

$$\text{Again, }2s^4\overset{\text{Gerretsen}}{\leq}2(4R^2+4Rr+3r^2)s^2\overset{?}{\leq}$$

$$(32R^2-16Rr+3r^2)s^2-r(128R^3+48R^2r-r^3)$$

$$\Leftrightarrow (24R^2 - 24Rr - 3r^2)s^2 \underbrace{\overset{?}{\geq}}_{(iii)} r(128R^3 + 48R^2r - r^3)$$

$$\text{Once again, LHS of (iii)} \overset{\text{Gerretsen}}{\geq} (24R^2 - 24Rr - 3r^2)(16Rr - 5r^2)$$

$$\overset{?}{\geq} r(128R^3 + 48R^2r - r^3) \Leftrightarrow 32t^3 - 69t^2 + 9t + 2 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(29t^2 + 3t(t-2) + (t-2) + 1) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \text{(iii)} \Rightarrow \text{(ii)}$$

$$\Rightarrow \text{(i) is true} \Rightarrow \sum_{cyc} \frac{h_a^2}{\sin^2 \frac{A}{2}} \geq \frac{2r}{R} \sum_{cyc} \frac{r_a^2}{\sin^2 \frac{A}{2}}$$

$$\forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1367. In any ΔABC, the following relationship holds :

$$\frac{(h_a^2 + w_b^2 + m_c^2)^5}{(r_a^5 + r_b^5)^2} + \frac{(h_b^2 + w_c^2 + m_a^2)^5}{(r_b^5 + r_c^5)^2} + \frac{(h_c^2 + w_a^2 + m_b^2)^5}{(r_c^5 + r_a^5)^2} \geq \frac{4 \cdot 6^6 \cdot r^{10}}{(81R^5 - 2560r^5)^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$r_a \leq \frac{a^2}{4r} \Rightarrow r_a^5 \leq s^4 \tan^4 \frac{A}{2} \cdot \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}{4r}$$

$$= \frac{4R^2s^4}{r}\left(\tan^2 \frac{A}{2}\right)\left(1 - \cos^2 \frac{A}{2}\right)^2 = \frac{4R^2s^4}{r}\left(\tan^2 \frac{A}{2}\right)\left(1 + \cos^4 \frac{A}{2} - 2\cos^2 \frac{A}{2}\right)$$

$$= \frac{4R^2s^4}{r}\left(\frac{r_a^2}{s^2} + \frac{\sin^2 A}{4} - (1 - \cos A)\right) \Rightarrow r_a^5 \leq \frac{4R^2s^4}{r}\left(\frac{r_a^2}{s^2} + \frac{a^2}{16R^2} - 1 + \cos A\right)$$

$$\text{and analogs} \Rightarrow \sum_{cyc} r_a^5 \leq \frac{4R^2s^4}{r}\left(\frac{(4R+r)^2 - 2s^2}{s^2} + \frac{\sum_{cyc} a^2}{16R^2} - 3 + 1 + \frac{r}{R}\right)$$

$$\overset{\text{Leibnitz}}{\leq} \frac{4R^2s^4}{r}\left(\frac{(4R+r)^2 - 4s^2}{s^2} + \frac{9}{16} + \frac{r}{R}\right)$$

$$= \frac{Rs^2}{4r}.\left(16R(4R+r)^2 - (55R - 16r)s^2\right) \overset{\text{Gerretsen}}{\leq}$$

$$\frac{Rs^2}{4r}.\left(16R(4R+r)^2 - (55R - 16r)(16Rr - 5r^2)\right)$$

$$\Rightarrow \sum_{cyc} r_a^5 \overset{(\bullet)}{\leq} \frac{Rs^2}{4r}.(256R^3 - 752R^2r + 547Rr^2 - 80r^3)$$

$$\text{Now, } \frac{(h_a^2 + w_b^2 + m_c^2)^5}{(r_a^5 + r_b^5)^2} + \frac{(h_b^2 + w_c^2 + m_a^2)^5}{(r_b^5 + r_c^5)^2} + \frac{(h_c^2 + w_a^2 + m_b^2)^5}{(r_c^5 + r_a^5)^2} =$$

$$\sum_{cyc}\frac{(h_a^2+w_b^2+m_c^2)^3}{\left(\frac{r_a^5+r_b^5}{h_a^2+w_b^2+m_c^2}\right)^2}\overset{\text{Holder}}{\geq}\frac{(\sum_{cyc}h_a^2+\sum_{cyc}w_b^2+\sum_{cyc}m_c^2)^3}{\left(\sum_{cyc}\frac{r_a^5+r_b^5}{h_a^2+w_b^2+m_c^2}\right)^2}\overset{w_a,m_a\geq h_a \text{ and analogs}}{\geq}\frac{27(\sum_{cyc}h_a^2)^5}{4(\sum_{cyc}r_a^5)^2}$$

$$\geq\frac{27\left(\frac{1}{3}(\sum_{cyc}h_a)^2\right)^5}{4(\sum_{cyc}r_a^5)^2}=\frac{(\sum_{cyc}h_a)^{10}}{36(\sum_{cyc}r_a^5)^2}\overset{?}{\geq}\frac{4*6^6*r^{10}}{(81R^5-2560r^5)^2}$$

$$\Leftrightarrow(81R^5-2560r^5)\left(\sum_{cyc}h_a\right)^5\underset{(*)}{\overset{?}{\geq}}12*216r^5\left(\sum_{cyc}r_a^5\right)$$

$$\text{We have}:(81R^5-2560r^5)\left(\sum_{cyc}h_a\right)^5$$

$$=(81R^5-2560r^5)\left(2rs\sum_{cyc}\frac{1}{a}\right)^4\left(\frac{s^2+4Rr+r^2}{2R}\right)\overset{\text{Bergstrom}}{\geq}$$

$$(81R^5-2560r^5)\left(2rs*\frac{9}{2s}\right)^4\left(\frac{s^2+4Rr+r^2}{2R}\right)\overset{?}{\geq}12*216r^5\left(\sum_{cyc}r_a^5\right)$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow}(81R^5-2560r^5)(s^2+4Rr+r^2)*81*81r^4$$

$$\overset{?}{\geq}24*216r^5*\frac{Rs^2}{4r}.(256R^3-752R^2r+547Rr^2-80r^3)$$

$$\Leftrightarrow(2465R^5+12032R^4r-8752R^3r^2+1280R^2r^3-207360r^5)s^2$$

$$+81(4Rr+r^2)(81R^5-2560r^5)\underset{(**)}{\overset{?}{\geq}}0$$

Case 1 $2465R^5+12032R^4r-8752R^3r^2+1280R^2r^3-207360r^5\geq 0$ *and then*:

$$\text{LHS of }(**)\geq 81(4Rr+r^2)(81R^5-2560r^5)$$

$$=81(4Rr+r^2)(81(R^5-32r^5)+32r^5)\overset{\text{Euler}}{\geq}81*32(4Rr+r^2)r^5>0$$

$\Rightarrow(**)$ **is true (strict inequality)**

Case 2 $2465R^5+12032R^4r-8752R^3r^2+1280R^2r^3-207360r^5<0$ *and then*:

$$\text{LHS of }(**)=-\left(-\left(\begin{matrix}2465R^5+12032R^4r-8752R^3r^2\\+1280R^2r^3-207360r^5\end{matrix}\right)\right)s^2$$

$$+81(4Rr+r^2)(81R^5-2560r^5)\overset{\text{Gerretsen}}{\geq}$$

$$-\left(-(2465R^5+12032R^4r-8752R^3r^2+1280R^2r^3-207360r^5)\right)\left(\begin{matrix}4R^2+\\4Rr+3r^2\end{matrix}\right)$$

$$+81(4Rr+r^2)(81R^5-2560r^5)\overset{?}{\geq}0$$

$$\Leftrightarrow 2465t^7+21058t^6+6769t^5+1552t^4-5284t^3-206400t^2$$

$$-414720t-207360\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(\begin{matrix}2465t^6+25988t^5+58745t^4+119042t^3\\+232800t^2+259200t+103680\end{matrix}\right)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow (**)$ is true $\therefore$ combining cases 1 and 2, $(**) \Rightarrow (**)$ is true $\forall \Delta ABC$

$$\Rightarrow \frac{\left(h_a^2+w_b^2+m_c^2\right)^5}{\left(r_a^5+r_b^5\right)^2}+\frac{\left(h_b^2+w_c^2+m_a^2\right)^5}{\left(r_b^5+r_c^5\right)^2}+\frac{\left(h_c^2+w_a^2+m_b^2\right)^5}{\left(r_c^5+r_a^5\right)^2}$$

$$\geq \frac{4*6^6*r^{10}}{(81R^5-2560r^5)^2}\ \forall \Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1368. **In ΔABC the following relationship holds**

$$n_a{}^2+2r_ah_a \geq r_br_c+m_bh_b+m_ch_c$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$n_a{}^2+2r_ah_a-r_br_c=\left(s(s-a)+\frac{s(b-c)^2}{a}\right)+\frac{4s^2r^2}{a(s-a)}-s(s-a)$$

$$=\frac{s(b-c)^2}{a}+\frac{4s(s-b)(s-c)}{a}=\frac{s(b-c)^2+s\left[a^2-(b-c)^2\right]}{a}=sa=\frac{sabc}{4F^2}.h_bh_c=\frac{R}{r}.h_bh_c.$$

Also, we have

$$\frac{R}{r}\geq\frac{m_b}{h_c}+\frac{m_c}{h_b}$$

(see, Bogdan Fuștei, Mohamed Amine Ben Ajiba, *New Triangle Inequalities With Brocard's Angle*, Lemma 12, 6 – 7, www.ssmrmh.ro)

Therefore

$$n_a{}^2+2r_ah_a=r_br_c+\frac{R}{r}.h_bh_c\geq r_br_c+\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)h_bh_c=r_br_c+m_bh_b+m_ch_c.$$

Equality holds iff ΔABC is equilateral.

1369. **In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{m_a^4}{a(1+\cos A)^4}\geq 9\sqrt{3}r^3$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

$$4m_a^2=2b^2+2c^2-a^2=(b^2+c^2-a^2)+b^2+c^2=2bc\cos A+(b^2+c^2)$$

$$\stackrel{AM-GM}{\geq} 2bc\cos A+2bc=2bc(1+\cos A)$$

$$\therefore 4m_a^2=2bc\,(1+\cos A)$$

$$m_a^4=\frac{1}{4}b^2c^2(1+\cos A)^2 \quad \text{(analog)}$$

$$\therefore \sum\frac{m_a^4}{a(1+\cos A)^4}\geq\frac{1}{4}\sum\frac{b^2c^2(1+\cos A)^2}{a(1+\cos A)^4}$$

$$=\frac{1}{4}\sum\frac{b^2c^2}{a}\cdot\frac{1}{(1+\cos A)^2}=\frac{1}{4}\sum\frac{b^2c^2}{a}\cdot\frac{1}{4\cos^4\frac{A}{2}}$$

$$\stackrel{AM-GM}{\geq}\frac{3}{4}\left[(abc)^3\cdot\frac{1}{64}\cdot\frac{1}{\prod\cos^3\frac{A}{2}\cdot\prod\cos\frac{A}{2}}\right]^{\frac{1}{3}}=\frac{3}{4}abc\cdot\frac{1}{4}\cdot\frac{1}{\prod\cos\frac{A}{2}}\left[\frac{1}{\prod\cos\frac{A}{2}}\right]^{\frac{1}{3}}$$

$$=\frac{3}{16}\cdot 4Rrs\cdot\frac{4R}{s}\left[\frac{4R}{s}\right]^{\frac{1}{3}}=\frac{3}{16}\times 16\,R^2r\left[\frac{4R}{s}\right]^{\frac{1}{3}}$$

$$\stackrel{Mitrinovic-Euler}{\geq} 3(2r)^2r\left[\frac{4r}{\frac{3\sqrt{3}R}{2}}\right]^{\frac{1}{3}}=3\cdot 4r^3\cdot\frac{2}{\sqrt{3}}=8\sqrt{3}r^3$$

Equality when $a=b=c$.

1370. If in Δ ABC, b = c then :

$$8\sin\frac{A}{2}\sin^2\frac{B}{2}\leq\sqrt{\cos\frac{A-B}{2}\cos\frac{A-C}{2}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

We will show $\cos\frac{B-C}{2}\geq\sqrt{\frac{2r}{R}}$ (1)

$$\cos\frac{B-C}{2}=\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}$$

Now $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$, $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}$

Using Ravi's transformation $a=y+z,\ b=z+x,\ c=x+y$

The inequality (1) is equivalent to $(2x+y+z)^2\geq 8x(y+z)$

which this is true by (AM-GM)

$$2s = 2(x + y + z)$$

$$\therefore s = x + y + z$$

$$\therefore \cos\frac{B-C}{2} \ge \sqrt{\frac{2r}{R}} \text{ (analog)}$$

$$\therefore \cos\frac{A-B}{2} \ge \sqrt{\frac{2r}{R}}, \cos\frac{A-C}{2} \ge \sqrt{\frac{2r}{R}}$$

[since $b = c \therefore B = C$]

$$\text{Now } \sqrt{\cos\frac{A-B}{2}\cos\frac{A-C}{2}} \ge \sqrt{\sqrt{\frac{2r}{R}} \cdot \sqrt{\frac{2r}{R}}} = \sqrt{\frac{2r}{R}}$$

$$8\sin\frac{A}{2} \cdot \sin^2\frac{B}{2} = 8\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{B}{2}$$

$$= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \quad (\because b = c, B = C) = 8 \cdot \frac{r}{4R} = \frac{2r}{R}$$

We need to show $\frac{2r}{R} \le \sqrt{\frac{2r}{R}}$ or $2Rr \ge 4r^2$ or $R \overset{Euler}{\ge} 2r$ (true)

Solution 2 by Aissa Hiyab-Morocco

$$b = c \Rightarrow B = C$$

$$\Rightarrow \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A-C}{2}\right) = \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A-C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$= \left(\frac{a+b}{c}\sin\frac{C}{2}\right)\left(\frac{a+c}{b}\sin\frac{B}{2}\right) \cdot \left(\frac{b+c}{a}\sin\frac{A}{2}\right) = \frac{(a+b)(a+c)(b+c)}{abc}\prod \sin\frac{A}{2}$$

$$= \frac{2s(s^2 + 2Rr + r^2)}{4Rrs} \times \left(\frac{r}{4R}\right) = \frac{s^2 + 2Rr + r^2}{8R^2}$$

$$8\sin\frac{A}{2}\sin^2\frac{B}{2} \overset{?}{\le} \sqrt{\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A-C}{2}\right)}$$

$$\Leftrightarrow 8\sin\frac{A}{2}\left(\sin\frac{B}{2}\sin\frac{C}{2}\right) \overset{?}{\le} \sqrt{\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A-C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$\Leftrightarrow 8 \times \frac{r}{4R} \overset{?}{\le} \sqrt{\frac{s^2 + 2Rr + r^2}{8R^2}} \Leftrightarrow 31r^2 - 2Rr \le s^2$$

$$31r^2 - 2Rr \overset{(*)}{\le} 16Rr - 5r^2 \overset{Gerretsen}{\le} 5$$

$$(*) \Rightarrow 31r^2 + 5r^2 \le 16Rr + 2Rr \Rightarrow 2r \le R \text{ true}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$b = c \Rightarrow B = C \therefore \sqrt{\cos\frac{A-B}{2}\cos\frac{A-C}{2}} - 8\sin\frac{A}{2}\sin^2\frac{B}{2}$$

$$= \sqrt{\cos\frac{A-B}{2}\cos\frac{A-B}{2}} - 4\sin\frac{B}{2}\left(\cos\frac{A-B}{2} - \sin\frac{B}{2}\right)$$

$$= \cos\frac{\pi-2B-B}{2} - 4\sin\frac{B}{2} * \left(\cos\frac{\pi-2B-B}{2} - \sin\frac{B}{2}\right)$$

$$= \sin\frac{3B}{2} - 4\sin\frac{B}{2} * \left(\sin\frac{3B}{2} - \sin\frac{B}{2}\right)$$

$$= 3\sin\frac{B}{2} - 4\sin^3\frac{B}{2} - 4\sin\frac{B}{2} * \left(2\sin\frac{B}{2} - 4\sin^3\frac{B}{2}\right)$$

$$= 3t - 4t^3 - 8t^2 + 16t^4 \left(t = \sin\frac{B}{2}\right) = t(4t+3)(2t-1)^2 \ge 0, '' ='' \text{ iff } t = \sin\frac{B}{2}$$

$$= \frac{1}{2} \Leftrightarrow \text{iff } \frac{B}{2} = \frac{C}{2} = \frac{\pi}{6} \Leftrightarrow A = B = C = \frac{\pi}{3}$$

$$\therefore 8\sin\frac{A}{2}\sin^2\frac{B}{2} \le \sqrt{\cos\frac{A-B}{2}\cos\frac{A-C}{2}} \ \forall \Delta ABC \,|\, b = c,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

1371. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(r_a+r_b)(r_c+r_b)}{ac} \le \frac{1}{16}\left(\sum_{cyc}\frac{bh_b}{(s-a)(s-c)}\right)^2 + \frac{9}{4}$$

Proposed by Gheorghe Molea – Romania

Solution 1 by Tapas Das-India

We have, $\frac{1}{r_b} + \frac{1}{r_c} = \frac{9}{F} = \frac{2}{h_a}$

$$\therefore r_b + r_c = \frac{2r_br_c}{h_a}$$

and $r_b + r_a = \frac{2r_br_c}{h_c}$

$$\therefore \sum_{cyc}\frac{(r_a+r_b)(r_b+r_c)}{ac} = \sum_{cyc}\frac{2r_br_a}{h_c}\cdot\frac{2r_br_c}{h_a}\cdot\frac{1}{ac} = \sum_{cyc}\frac{4\cdot r_ar_br_c}{h_ah_c\cdot ac}\cdot r_b$$

$$=\sum_{cyc}\frac{4\cdot s^2r}{2F\cdot 2F}r_b=\sum_{cyc}\frac{r_bs^2r}{s^2r^2}=\frac{1}{r}\sum r_b=\frac{4R+r}{r}\quad (1)$$

Now $(s-a)=\frac{F}{r_a},(s-c)=\frac{F}{r_c}$

$$\therefore\left(\sum_{cyc}\frac{bh_b}{(s-a)(s-c)}\right)^2=\left(\sum_{cyc}\frac{bh_b}{F\cdot F}r_ar_c\right)^2=\left(\sum_{cyc}\frac{\frac{2F}{b}\cdot b}{F^2}r_ar_c\right)^2$$

$$=\frac{4}{F^2}\left(\sum r_ar_c\right)^2=\frac{4}{F^2}\cdot s^9=\frac{4S^9}{r^2s^2}=\frac{4s^2}{r^2}$$

$$\therefore\frac{1}{16}\left(\sum\frac{bh_b}{(s-a)(s-c)}\right)^2+\frac{9}{4}=\frac{1}{16}\cdot\frac{4s^2}{r^2}+\frac{9}{4}=\frac{1}{4}\frac{s^2}{r^2}+\frac{9}{4}\quad (2)$$

From (1) and (2) we need to show $\frac{1}{4}\cdot\frac{s^2}{r^2}+\frac{9}{4}\ge\frac{4R+r}{r}$

or $s^2+9r^2\ge 16Rr+4r^2$ or $s^2\ge 16Rr-5r^2$ (True)

(Gerretsen's)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

1) $ac=\Delta^2\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\cdot\left(\frac{1}{r_b}+\frac{1}{r_c}\right)=$

$$=\frac{\Delta^2(r_a+r_b)(r_b+r_c)}{r_b(r_ar_br_c)}=\frac{r(r_ar_br_c)(r_a+r_b)r_ar_br_c}{r_b(r_ar_br_c)}=\frac{r(r_a+r_b)(r_b+r_c)}{r_b}=ac\qquad (*)$$

$$\sum\frac{(r_a+r_b)(r_b+r_c)}{\frac{r(r_a+r_b)(r_b+r_c)}{r_b}}=\sum\frac{r_b}{r}=\frac{4R+r}{r}\quad (1)$$

$$LHS=\frac{4R+r}{r}\quad (1)$$

2) $\frac{1}{16}\left(\sum\frac{2\Delta}{(p-a)(p-b)}\right)+\frac{9}{4}=\frac{1}{4}\left(\sum\sqrt{\frac{p(p-c)}{(p-a)(p-b)}}\right)+\frac{9}{4}=\frac{1}{4}\left(\sum\cot\frac{c}{2}\right)^2+\frac{9}{4}=$

$$=\frac{1}{4}\left(\sum\frac{p}{r_a}\right)^2+\frac{9}{4}=\frac{p^2}{4}\left(\frac{\sum r_ar_b}{r_ar_br_c}\right)^2+\frac{9}{4}=\frac{p^2}{4}\left(\frac{p^2}{p^2r}\right)^2+\frac{9}{4}=\frac{p^2}{4r^2}+\frac{9}{4}\qquad (**)$$

$$(1);(**)\Rightarrow\frac{4R+r}{r}\overset{?}{\le}\frac{p^2}{4r^2}+\frac{9}{4}$$

$$p^2+9r^2\overset{?}{\ge}4r(4R+r)$$

$$p^2+9r^2\ge 16Rr-5r^2+9r62=16Rr+4r^2$$

1372. In ΔABC the following relationship holds:

$$a^2\cdot\sqrt[3]{bc}+b^2\cdot\sqrt[3]{ca}+c^2\cdot\sqrt[3]{ab}\le R^2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

$$a^2\cdot\sqrt[3]{bc}+b^2\cdot\sqrt[3]{ca}+c^2\cdot\sqrt[3]{ab}$$

$$\text{WLOG } a\ge b\ge c$$

$$\therefore ab\ge ac\ge bc$$

$$\therefore a^2\sqrt[3]{bc}+b^2\sqrt[3]{ca}+c^2\sqrt[3]{ab}\overset{Chebysev}{\le}\frac{1}{3}(a^2+b^2+c^2)\left(\sqrt[3]{bc}+\sqrt[3]{ca}+\sqrt[3]{ab}\right)$$

$$\le\frac{1}{3}(a^2+b^2+c^2)\frac{\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}\right)^2}{3}$$

$$[\text{Note: }(\sum x)^2\ge 3(\sum xy)]$$

$$\forall x,y,z>0$$

$$\overset{Leibniz}{\le}\frac{1}{3}\cdot 9R^2\frac{\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}\right)^2}{3}=R^2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^2$$

1373. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}\ge 3\sum_{cyc}\frac{h_a^2}{\cos^2\frac{A}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}=\sum_{cyc}\left(h_a^2\left(1+\cot^2\frac{A}{2}\right)\right)=\sum_{cyc}h_a^2+s^2\sum_{cyc}\frac{h_a^2}{r_a^2}$$

$$=\sum_{cyc}\frac{a^2b^2}{4R^2}+\frac{s^2}{r^2s^4}\sum_{cyc}h_a^2r_b^2r_c^2=\frac{1}{4R^2}\sum_{cyc}a^2b^2+\frac{4r^2s^4}{r^2s^2}\sum_{cyc}\frac{(s-a)^2}{a^2}$$

$$=\frac{1}{4R^2}\sum_{cyc}a^2b^2+4s^2\left(s^2\sum_{cyc}\frac{1}{a^2}-2s\sum_{cyc}\frac{1}{a}+3\right)$$

$$=\frac{1}{4R^2}\sum_{cyc}a^2b^2+\frac{4s^4}{16R^2r^2s^2}\sum_{cyc}a^2b^2-\frac{8s^3}{4Rrs}\sum_{cyc}ab+12s^2$$

$$=\frac{(s^2+r^2)\left((s^2+4Rr+r^2)-16Rrs^2\right)-8Rrs^2(s^2+4Rr+r^2)+48R^2r^2s^2}{4R^2r^2}$$

$$= \frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{4R^2r^2} \overset{(*)}{=} \sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}$$

$$\text{Again}, \sum_{cyc}\frac{h_a^2}{\cos^2\frac{A}{2}} = \sum_{cyc}h_a^2+\sum_{cyc}h_a^2\tan^2\frac{A}{2} = \frac{\sum_{cyc}a^2b^2}{4R^2}+4r^2\sum_{cyc}\frac{r_a^2}{a^2}$$

$$= \frac{\sum_{cyc}a^2b^2}{4R^2}+4r^2\left(\left(\sum_{cyc}\frac{r_a}{a}\right)^2-2\sum_{cyc}\frac{r_br_c}{bc}\right)$$

$$= \frac{\sum_{cyc}a^2b^2}{4R^2}+4r^2\left(\left(\sum_{cyc}\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}\right)^2-2\sum_{cyc}\frac{s(s-a)}{bc}\right)$$

$$= \frac{\sum_{cyc}a^2b^2}{4R^2}+4r^2\left(\left(\frac{s}{4R}.\frac{s^2+(4R+r)^2}{s^2}\right)^2-\sum_{cyc}(1+\cos A)\right)$$

$$= \frac{(s^2+4Rr+r^2)-16Rrs^2}{4R^2}+4r^2\left(\frac{(s^2+(4R+r)^2)^2}{16R^2s^2}-\frac{4R+r}{R}\right)$$

$$= \frac{s^6-(8Rr-3r^2)s^4-r^2s^2(16R^2-8Rr-3r^2)+r^2(4R+r)^4}{4R^2s^2} \overset{(**)}{=} \sum_{cyc}\frac{h_a^2}{\cos^2\frac{A}{2}}$$

$$\therefore \text{via } (*),(**), \sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}} \ge 3\sum_{cyc}\frac{h_a^2}{\cos^2\frac{A}{2}}$$

$$\Leftrightarrow \frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{4R^2r^2}$$

$$\ge 3.\frac{s^6-(8Rr-3r^2)s^4-r^2s^2(16R^2-8Rr-3r^2)+r^2(4R+r)^4}{4R^2s^2}$$

$$\Leftrightarrow s^8-16Rrs^6+r^2s^4(32R^2+16Rr-6r^2)$$

$$+r^4s^2(64R^2-16Rr-8r^2)-3r^4(4R+r)^4 \overset{(i)}{\ge} 0$$

$$\text{Now, LHS of (i)} \overset{\text{Gerretsen}}{\ge} (16Rr-5r^2)s^6-16Rrs^6+r^2s^4(32R^2+16Rr-6r^2)$$

$$+r^4s^2(64R^2-16Rr-8r^2)-3r^4(4R+r)^4 \overset{?}{\ge} 0$$

$$\Leftrightarrow (32R^2+16Rr-6r^2)s^4+r^2s^2(64R^2-16Rr-8r^2)-3r^2(4R+r)^4 \underset{(ii)}{\overset{?}{\ge}} 5s^6$$

$$\text{Again}, 5s^6 \overset{\text{Gerretsen}}{\le} 5(4R^2+4Rr+3r^2)s^4 \overset{?}{\le} \text{LHS of (ii)}$$

$$\Leftrightarrow (12R^2-4Rr-21r^2)s^4+r^2(64R^2-16Rr-8r^2)s^2-3r^2(4R+r)^4 \underset{(iii)}{\overset{?}{\ge}} 0$$

$$\text{Once again, LHS of (iii)} \overset{\text{Gerretsen}}{\ge} (12R^2-4Rr-21r^2)(16Rr-5r^2)s^2$$

$$+r^2(64R^2-16Rr-8r^2)s^2-3r^2(4R+r)^4$$

$$= r(192R^3-60R^2r-332Rr^2+97r^3)s^2-3r^2(4R+r)^4$$

$$\overset{Gerretsen}{\geq} r(192R^3-60R^2r-332Rr^2+97r^3)(16Rr-5r^2)-3r^2(4R+r)^4$$

$$\left(\begin{array}{c}\because 192R^3-60R^2r-332Rr^2+97r^3=\\(R-2r)(192R^2+324Rr+316r^2)+729r^3\overset{Euler}{\geq}729r^3>0\end{array}\right)$$

$$\overset{?}{\geq}0\Leftrightarrow 576t^4-672t^3-1325t^2+791t-122\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)(576t^3+297t^2+183t(t-2)+t+61)\right)\overset{?}{\geq}0\to true\because t\overset{Euler}{\geq}2$$

$$\therefore (iii)\Rightarrow(ii)\Rightarrow(i)\text{ is true}\Rightarrow\sum_{cyc}\frac{h_a^2}{\sin^2\frac{A}{2}}\geq 3\sum_{cyc}\frac{h_a^2}{\cos^2\frac{A}{2}}$$

$\forall\,\Delta ABC$,'' ='' iff ΔABC is equilateral (QED)

1374. In ΔABC the following relationship holds:

$$\frac{a}{s}\cdot\left(\frac{sh_a}{bc}\right)^n+\frac{b}{s}\cdot\left(\frac{sh_b}{ca}\right)^n+\frac{c}{s}\cdot\left(\frac{sh_c}{ab}\right)^n\geq\frac{3^n\sqrt{3^n}}{2^{n-1}}\cdot\left(\frac{r}{R}\right)^n,n\in\mathbb{N}^*$$

Proposed by Emil Popa-Romania

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a}{s}\cdot\left(\frac{sh_a}{bc}\right)^n=\sum_{cyc}\frac{a}{s}\cdot\left(\frac{sah_a}{abc}\right)^n=\frac{1}{s}\cdot\left(\frac{s}{abc}\right)^n\sum_{cyc}a\cdot(ah_a)^n=$$

$$=\frac{s^{n-1}}{(abc)^n}\cdot\sum_{cyc}a\cdot(2F)^n=\frac{s^{n-1}\cdot(2F)^n}{(abc)^n}\cdot\sum_{cyc}a=$$

$$=\frac{s^{n-1}\cdot(2F)^n}{(abc)^n}\cdot 2s=\frac{s^{n-1}\cdot(2F)^n}{(4RF)^n}\cdot 2s=\frac{s^{n-1}\cdot 2^n\cdot 2s}{4^n\cdot R^n}=$$

$$=\frac{2^{n+1}}{2^{2n}}\cdot\frac{s^n}{R^n}\overset{MITRINOVIC}{\geq}\frac{1}{2^{n-1}}\cdot\frac{(3\sqrt{3}r)^n}{R^n}=\frac{3^n\sqrt{3^n}}{2^{n-1}}\cdot\left(\frac{r}{R}\right)^n$$

Equality holds for $a=b=c$.

1375. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\cdot\left(\frac{r_a}{h_a}\right)^2+\frac{y+z}{x}\cdot\left(\frac{r_b}{h_b}\right)^2+\frac{z+x}{y}\cdot\left(\frac{r_c}{h_c}\right)^2\geq 6$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{x+y}{z}\cdot\left(\frac{r_a}{h_a}\right)^2+\frac{y+z}{x}\cdot\left(\frac{r_b}{h_b}\right)^2+\frac{z+x}{y}\cdot\left(\frac{r_c}{h_c}\right)^2\overset{AM-GM}{\geq}$$

$$\geq\frac{2\sqrt{xy}}{z}\cdot\left(\frac{r_a}{h_a}\right)^2+\frac{2\sqrt{yz}}{x}\cdot\left(\frac{r_b}{h_b}\right)^2+\frac{2\sqrt{zx}}{y}\cdot\left(\frac{r_c}{h_c}\right)^2\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{2\sqrt{xy}}{z}\cdot\left(\frac{r_a}{h_a}\right)^2\cdot\frac{2\sqrt{yz}}{x}\cdot\left(\frac{r_b}{h_b}\right)^2\cdot\frac{2\sqrt{zx}}{y}\cdot\left(\frac{r_c}{h_c}\right)^2}=$$

$$=3\sqrt[3]{8\left(\frac{r_a}{h_a}\right)^2\left(\frac{r_b}{h_b}\right)^2\left(\frac{r_c}{h_c}\right)^2}=6\sqrt[3]{\left(\frac{r_a}{h_a}\right)^2\left(\frac{r_b}{h_b}\right)^2\left(\frac{r_c}{h_c}\right)^2}\geq 6\Leftrightarrow$$

$$\left(\frac{r_a}{h_a}\right)^2\left(\frac{r_b}{h_b}\right)^2\left(\frac{r_c}{h_c}\right)^2\geq 1\Leftrightarrow r_ar_br_c\geq h_ah_bh_c\Leftrightarrow$$

$$\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\geq\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\Leftrightarrow$$

$$abc\geq 8(s-a)(s-b)(s-c)\Leftrightarrow$$

$$(b+c-a)(c+a-b)(a+b-c)\leq abc\ \textit{(Padoa's inequality)}$$

Equality holds for $a=b=c,\ x=y=z$.

Note:

$$(b+c-a)(c+a-b)(a+b-c)=\prod_{cyc}\sqrt{(b+c-a)(c+a-b)}\overset{AM-GM}{\leq}$$

$$\leq\prod_{cyc}\frac{(b+c-a)+(c+a-b)}{2}=abc$$

1376. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\cdot\left(\frac{a}{r_a}\right)^2+\frac{y+z}{x}\cdot\left(\frac{b}{r_b}\right)^2+\frac{z+x}{y}\cdot\left(\frac{c}{r_c}\right)^2\geq 216\left(\frac{r}{s}\right)^2$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{x+y}{z}\cdot\left(\frac{a}{r_a}\right)^2+\frac{y+z}{x}\cdot\left(\frac{b}{r_b}\right)^2+\frac{z+x}{y}\cdot\left(\frac{c}{r_c}\right)^2\overset{AM-GM}{\geq}$$

$$\geq \frac{2\sqrt{xy}}{z}\cdot\left(\frac{a}{r_a}\right)^2+\frac{2\sqrt{yz}}{x}\cdot\left(\frac{b}{r_b}\right)^2+\frac{2\sqrt{zx}}{y}\cdot\left(\frac{c}{r_c}\right)^2 \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{2\sqrt{xy}}{z}\cdot\left(\frac{a}{r_a}\right)^2\cdot\frac{2\sqrt{yz}}{x}\cdot\left(\frac{b}{r_b}\right)^2\cdot\frac{2\sqrt{zx}}{y}\cdot\left(\frac{c}{r_c}\right)^2}=$$

$$=3\sqrt[3]{8\left(\frac{a}{r_a}\right)^2\left(\frac{b}{r_b}\right)^2\left(\frac{c}{r_c}\right)^2}=6\sqrt[3]{\left(\frac{a}{r_a}\right)^2\left(\frac{b}{r_b}\right)^2\left(\frac{c}{r_c}\right)^2}=$$

$$=6\sqrt[3]{\left(\frac{abc}{r_ar_br_c}\right)^2}=6\sqrt[3]{\left(\frac{4Rrs}{rs^2}\right)^2}=6\sqrt[3]{\left(\frac{4R}{s}\right)^2}\overset{MITRINOVIC}{\geq}$$

$$\geq 6\sqrt[3]{\left(\frac{4\cdot\frac{2s}{3\sqrt{3}}}{s}\right)^2}=6\sqrt[3]{\left(\frac{4}{3}\right)^3}=6\cdot\frac{4}{3}=8\geq 216\left(\frac{r}{s}\right)^2\Leftrightarrow$$

$$\Leftrightarrow s^2\geq 27r^2\Leftrightarrow s\geq 3\sqrt{3}r \text{ (Mitrinovic).}$$

Equality holds for: $a=b=c, x=y=z$.

1377. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{h_bh_c}{(r-h_b)(r-h_c)}\leq\sum_{cyc}\frac{r_br_c}{(r-r_b)(r-r_c)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_bh_c}{(r-h_b)(r-h_c)}=\sum_{cyc}\frac{\frac{4r^2s^2}{bc}}{r^2\left(\frac{2s}{b}-1\right)\left(\frac{2s}{c}-1\right)}=\sum_{cyc}\frac{4s^2(b+c)}{(c+a)(a+b)(b+c)}$$

$$=\frac{4s^2.4s}{2s(s^2+2Rr+r^2)}\Rightarrow\sum_{cyc}\frac{h_bh_c}{(r-h_b)(r-h_c)}\overset{(i)}{=}\frac{8s^2}{s^2+2Rr+r^2}$$

$$\sum_{cyc}\frac{r_br_c}{(r-r_b)(r-r_c)}=\sum_{cyc}\frac{\frac{r^2s^2}{(s-b)(s-c)}}{r^2\left(\frac{s}{s-b}-1\right)\left(\frac{s}{s-c}-1\right)}=\sum_{cyc}\frac{s^2a}{4Rrs}=\frac{s^2*2s}{4Rrs}$$

$$\Rightarrow\sum_{cyc}\frac{r_br_c}{(r-r_b)(r-r_c)}\overset{(ii)}{=}\frac{s^2}{2Rr}\therefore (i),(ii)\Rightarrow$$

$$\sum_{cyc}\frac{h_bh_c}{(r-h_b)(r-h_c)}\le\sum_{cyc}\frac{r_br_c}{(r-r_b)(r-r_c)}\Leftrightarrow\frac{8s^2}{s^2+2Rr+r^2}\le\frac{s^2}{2Rr}$$

$$\Leftrightarrow s^2+2Rr+r^2\ge 16Rr\Leftrightarrow s^2-16Rr+5r^2+2r(R-2r)\ge 0\to\text{true}$$

$$\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0 \text{ and } 2r(R-2r)\overset{\text{Euler}}{\ge}0\therefore\sum_{cyc}\frac{h_bh_c}{(r-h_b)(r-h_c)}$$

$$\le\sum_{cyc}\frac{r_br_c}{(r-r_b)(r-r_c)}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1378. In ΔABC the following relationship holds:

$$\left(\frac{r_a}{h_a}\right)^2+\left(\frac{r_b}{h_b}\right)^2+\left(\frac{r_c}{h_c}\right)^2\ge 1+\frac{4r}{R}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\left(\frac{r_a}{h_a}\right)^2+\left(\frac{r_b}{h_b}\right)^2+\left(\frac{r_c}{h_c}\right)^2\overset{AM-GM}{\ge}3\sqrt[3]{\left(\frac{r_a}{h_a}\right)^2\cdot\left(\frac{r_b}{h_b}\right)^2\cdot\left(\frac{r_c}{h_c}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{r_ar_br_c}{h_ah_bh_c}\right)^2}\ge 3\sqrt[3]{\left(\frac{h_ah_bh_c}{h_ah_bh_c}\right)^2}=3=1+2=1+4\cdot\frac{1}{2}=1+4\cdot\frac{r}{R}$$

Equality holds for $a=b=c$.

1379. In any Δ ABC, the following relationship holds :

$$\frac{2Rm_a}{r}\ge(r_b+r_c)\left(\sqrt{\frac{m_a}{h_b}}+\sqrt{\frac{m_b}{h_a}}\right)$$

Proposed by Bogdan Fuştei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\overset{?}{\ge}32Rr\cos^2\frac{A}{2}\overset{\text{via (i)}}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0 \Leftrightarrow (b+c-2a)^2\overset{?}{\geq}0\to \text{true}$$

$$\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\Rightarrow\frac{2Rm_a}{r(r_b+r_c)}\overset{\text{Lascu and (i)}}{\geq}\frac{2R*\frac{b+c}{2}\cos\frac{A}{2}}{r*4R\cos^2\frac{A}{2}}$$

$$\geq\frac{2R*\sqrt{8Rr}.\cos\frac{A}{2}*\cos\frac{A}{2}}{r*4R\cos^2\frac{A}{2}}=\sqrt{\frac{2R}{r}}\therefore\frac{2Rm_a}{r(r_b+r_c)}\overset{(*)}{\geq}\sqrt{\frac{2R}{r}}$$

$$\text{Again, }\sqrt{\frac{m_a}{h_b}}+\sqrt{\frac{m_b}{h_a}}\overset{\text{CBS}}{\leq}\sqrt{2}*\sqrt{\frac{m_a}{h_b}+\frac{m_b}{h_a}}\leq\sqrt{\frac{2R}{r}}$$

$\left(\because \frac{R}{r}\geq\frac{m_a}{h_b}+\frac{m_b}{h_a}\right.$ reference : article titled "New Triangle Inequalities With Brocard's Angle" by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 – 7, published at : www.ssmrmh.ro$\left.\right)$

$$\therefore\sqrt{\frac{m_a}{h_b}}+\sqrt{\frac{m_b}{h_a}}\overset{(**)}{\leq}\sqrt{\frac{2R}{r}}\therefore \text{combining cases } (*) \text{ and } (**), \frac{2Rm_a}{r(r_b+r_c)}\geq\sqrt{\frac{m_a}{h_b}}+\sqrt{\frac{m_b}{h_a}}$$

$$\Rightarrow\frac{2Rm_a}{r}\geq(r_b+r_c)\left(\sqrt{\frac{m_a}{h_b}}+\sqrt{\frac{m_b}{h_a}}\right)\ \forall\,\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1380. In any Δ ABC, the following relationship holds :

$$\frac{r_a^5}{r_a^3r_br_c+1}+\frac{r_b^5}{r_b^3r_cr_a+1}+\frac{r_c^5}{r_c^3r_ar_b+1}\geq\frac{81r^7r_a^2\text{ctg}^2\frac{A}{2}}{r^4r_a^3r_br_c\text{ctg}^2\frac{A}{2}+r(r_ar_br_c-24r^3)}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{r_a^5}{r_a^3r_br_c+1}+\frac{r_b^5}{r_b^3r_cr_a+1}+\frac{r_c^5}{r_c^3r_ar_b+1}\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}r_a\right)^5}{27\left(rs^2\left(\sum_{cyc}r_a^2\right)+3\right)}\overset{?}{\geq}$$

$$\frac{81r^7r_a^2\text{ctg}^2\frac{A}{2}}{r^4r_a^3r_br_c\text{ctg}^2\frac{A}{2}+r(r_ar_br_c-24r^3)}=\frac{81r^7*s^2\tan^2\frac{A}{2}*\text{ctg}^2\frac{A}{2}}{r^4*rs^2*s^2\tan^2\frac{A}{2}*\text{ctg}^2\frac{A}{2}+r^2(s^2-24r^2)}$$

$$\Leftrightarrow \boxed{\left(\sum_{cyc}r_a\right)^5(s^2-24r^2)+\left(\sum_{cyc}r_a\right)^5*r^3s^4\underset{(*)}{\overset{?}{\geq}}2187r^6s^4\left(\sum_{cyc}r_a^2\right)+81^2r^5s^2}$$

$$\text{Now, }\left(\sum_{cyc}r_a\right)^5*r^3s^4\overset{A-G}{\geq}27rs^2\left(\sum_{cyc}r_a\right)^2*r^3s^4\overset{?}{\geq}2187r^6s^4\left(\sum_{cyc}r_a^2\right)$$

$$\Leftrightarrow s^2(4R+r)^2 \overset{?}{\geq} 81r^2\left((4R+r)^2-2s^2\right)$$

$$\Leftrightarrow s^2\left((4R+r)^2+162r^2\right) \underset{(1)}{\overset{?}{\geq}} 81r^2(4R+r)^2$$

We have : LHS of (1) $\overset{\text{Gerretsen}}{\geq} (16Rr-5r^2)\left((4R+r)^2+162r^2\right) \overset{?}{\geq} 81r^2(4R+r)^2$

$$\Leftrightarrow 8t^3-39t^2+60t-28 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (8t-7)(t-2)^2 \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (1) \text{ is true} \therefore \boxed{\left(\sum_{cyc} r_a\right)^5 * r^3s^4 \overset{(\bullet)}{\geq} 2187r^6s^4\left(\sum_{cyc} r_a^2\right)}$$

$$\text{Again, } \left(\sum_{cyc} r_a\right)^5 (s^2-24r^2) \overset{A-G}{\geq} 27rs^2(4R+r)^2(s^2-24r^2) \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\geq}$$

$$27rs^2(9r)^2 * 3r^2 = 81^2r^5s^2 \therefore \boxed{\left(\sum_{cyc} r_a\right)^5 (s^2-24r^2) \overset{(\bullet\bullet)}{\geq} 81^2r^5s^2} \therefore (\bullet)+(\bullet\bullet)$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{r_a^5}{r_a^3r_br_c+1}+\frac{r_b^5}{r_b^3r_cr_a+1}+\frac{r_c^5}{r_c^3r_ar_b+1}$$

$$\geq \frac{81r^7r_a^2\text{ctg}^2\frac{A}{2}}{r^4r_a^3r_br_c\text{ctg}^2\frac{A}{2}+r(r_ar_br_c-24r^3)} \; \forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sum_{cyc}\frac{{r_a}^5}{{r_a}^3r_br_c+1}=\sum_{cyc}\frac{{r_a}^2}{r_br_c+\frac{1}{{r_a}^3}} \geq \frac{(r_a+r_b+r_c)^2}{\sum_{cyc}\left(r_br_c+\frac{1}{{r_a}^3}\right)}$$

$$=\frac{(4R+r)^2}{s^2+\left(\sum_{cyc}\frac{1}{r_a}\right)^3-3\prod_{cyc}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)}$$

$$\overset{Euler}{\geq} \frac{(4.2r+r)^2}{s^2+\left(\frac{1}{r}\right)^3-3.\frac{4Rs^2}{(s^2r)^2}} = \frac{81s^2r^7}{s^4r^5+s^2r^2-12Rr^3} \overset{Euler}{\geq} \frac{81r^7s^2}{s^4r^5+r(s^2r-24r^3)},$$

and since $r_ar_br_c = s^2r$ **and** $r_a\cot\frac{A}{2}=s$, **then**

$$\sum_{cyc}\frac{{r_a}^5}{{r_a}^3r_br_c+1} \geq \frac{81r^7{r_a}^2\cot^2\frac{A}{2}}{r^4{r_a}^3r_br_c\cot^2\frac{A}{2}+r(r_ar_br_c-24r^3)}.$$

Equality holds iff ΔABC **is equilateral.**

1381. In acute ΔABC holds:

$$\sin^4 A\,(1-\sin A)^5+\sin^4 B\,(1-\sin B)^5+\sin^4 C\,(1-\sin C)<\frac{3}{256}$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das-India

$$\sin^4 A\,(1-\sin A)^5=(1-\sin A)[\sin A\,(1-\sin A)]^4$$

$$\overset{AM-GM}{\leq}(1-\sin A)\left[\frac{\sin A+1-\sin A}{2}\right]^8=(1-\sin A)\cdot\frac{1}{256}\text{ (analog)}$$

$$\therefore\sum\sin^4 A\,(1-\sin A)^5\leq\sum(1-\sin A)\cdot\frac{1}{256}<\sum\frac{1}{256}=\frac{3}{256}(\because\sin A<1)$$

1382.

If $x,y,z>0$, then, in any Δ ABC, the following relationship holds:

$$\frac{x}{y+z}.m_a^2+\frac{y}{z+x}.m_b^2+\frac{z}{x+y}.m_c^2\geq\frac{16Fs-27R^3}{8R}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution 1 by Tapas Das-India

Let us consider m_a,m_b,m_c as sides of a triangle and

F' = Area of the new triangle

$$\therefore 16F'^2=2\sum(m_a^2m_b^2)-\sum m_a^4$$

$$16F'^2=2\sum\frac{1}{4}(2b^2+2c^2-a^2)(2a^2+2c^2-b^2)-\sum\frac{1}{16}(2b^2+2c^2-a^2)^2$$

$$16F'^2=9F^2,\qquad F'=\frac{3}{4}F$$

$$\therefore\frac{x}{y+z}m_a^2+\frac{y}{z+x}m_b^2+\frac{z}{x+y}m_c^2\overset{Tsintifas}{\geq}2\sqrt{3}F'=2\sqrt{3}\cdot\frac{3}{4}F$$

$$=\frac{6\sqrt{3}F}{4}=\frac{6\sqrt{3}Fs}{4s}\geq\frac{6\sqrt{3}Fs}{4\cdot\frac{3\sqrt{3}R}{2}}=\frac{Fs}{R}=\frac{8Fs}{8R}=\frac{16Fs-8Fs}{8R}=\frac{16Fs-8rs^2}{8R}$$

$$\overset{Mitrinovic-Euler}{\geq}\frac{16Fs-8\cdot\frac{R}{2}\cdot\frac{27R^2}{4}}{8R}=\frac{16Fs-27R^3}{8R}$$

Solution 2 by Tapas Das-India

Let $x+y=a', x+z=b', z+y=c'$

$$\therefore x+y+z=\frac{1}{2}(a'+b'+c')$$

$$\therefore x=\frac{1}{2}(a'+b'+c')-c'=\frac{a'+b'-c'}{2}$$

$$\therefore \frac{x}{y+z}=\frac{1}{2}\left(\frac{a'}{c'}+\frac{b'}{c'}-1\right)$$

$$(m_a^2\cdot m_b^2\cdot m_c^2)^{\frac{1}{3}}\geq[s(s-a)(s)(s-b)s(s-c)]^{\frac{1}{3}}=$$

$$\left(m_a\geq\sqrt{s(s-a)}\right)=[s^2F^2]^{\frac{1}{3}}=\left[\frac{s^3F}{sF}\right]^{\frac{1}{3}}=sF\left[\frac{1}{sF}\right]^{\frac{1}{3}}$$

$$=sF\left[\frac{1}{r\cdot s^2}\right]^{\frac{1}{3}}\overset{Euler-Mitrinovic}{\geq}\frac{Fs}{\left(\frac{R}{2}\cdot\frac{27}{4}R^2\right)^{\frac{1}{3}}}=\frac{2FS}{3R}$$

$$\therefore\sum\frac{x}{y+z}\cdot m_a^2=\frac{1}{2}\sum\left(\frac{a'}{c'}+\frac{b'}{c'}-1\right)m_a^2$$

$$=\frac{1}{2}\left[\left(\frac{a'}{c'}m_a^2+\frac{c'}{b'}m_b^2+\frac{b'}{a'}m_c^2\right)+\left(\frac{b'}{c'}m_a^2+\frac{a'}{b'}m_b^2+\frac{c'}{a'}m_c^2\right)-\sum(m_a^2)\right]$$

$$\overset{AM-GM}{\geq}\frac{1}{2}\left[3(m_a^2m_b^2m_c^2)^{\frac{1}{3}}\times 2-\frac{3}{4}\sum a^2\right]$$

$$\overset{Leibniz}{\geq}\frac{1}{2}\left[2\times3\times\frac{2FS}{3R}-\frac{3}{4}9R^2\right]=\frac{1}{2}\left[2\times\frac{2FS}{R}-\frac{27R^2}{4}\right]=\frac{16Fs-27R^3}{8R}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{y+z}.m_a^2+\frac{y}{z+x}.m_b^2+\frac{z}{x+y}.m_c^2=\sum_{cyc}\left(\frac{\sum_{cyc}x-(y+z)}{y+z}.m_a^2\right)$$

$$=\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{m_a^2}{y+z}\right)-\sum_{cyc}m_a^2\overset{Bergstrom}{\geq}\left(\frac{\sum_{cyc}x}{2\sum_{cyc}x}\right)\left(\sum_{cyc}m_a\right)^2-\sum_{cyc}m_a^2$$

$$=\sum_{cyc} m_a m_b - \frac{\sum_{cyc} m_a^2}{2} \overset{\text{Tereshin}}{\geq} \sum_{cyc} \frac{(b^2+c^2)(c^2+a^2)}{16R^2} - \frac{1}{2} * \frac{3}{4} * \sum_{cyc} a^2 \overset{\substack{\text{Reverse CBS}\\+\\\text{Leibnitz}}}{\geq}$$

$$\sum_{cyc} \frac{(bc+ca)^2}{16R^2} - \frac{27R^2}{8} \overset{A-G}{\geq} \sum_{cyc} \frac{c^2 * 4ab}{16R^2} - \frac{27R^2}{8} = \frac{4 * 4Rrs * 2s}{16R^2} - \frac{27R^2}{8}$$

$$= \frac{16Fs - 27R^3}{8R}, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1383. In ΔABC holds :

$$\frac{w_a}{h_a}\left(\sqrt{\frac{m_b}{h_c}} + \sqrt{\frac{m_c}{h_b}}\right) \leq \frac{R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{w_a}{h_a} = \frac{2\sqrt{bc.s(s-a)}}{b+c} \cdot \frac{2R}{bc} = \frac{4R\sqrt{s(s-a)}}{[2(s-a)+a]\sqrt{bc}} \overset{AM-GM}{\leq}$$

$$\leq \frac{4R\sqrt{s(s-a)}}{2\sqrt{2(s-a)a.bc}} = 2R\sqrt{\frac{s}{8Rsr}} = \sqrt{\frac{R}{2r}}.$$

Now we will use the following inequality

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \frac{R}{r}$$

(see, Bogdan Fuștei, Mohamed Amine Ben Ajiba, *New Triangle Inequalities With Brocard's Angle*, Lemma 12, 6 − 7, www.ssmrmh.ro) :

$$\sqrt{\frac{m_b}{h_c}} + \sqrt{\frac{m_c}{h_b}} \leq \sqrt{2\left(\frac{m_b}{h_c} + \frac{m_c}{h_b}\right)} \leq \sqrt{\frac{2R}{r}}.$$

Therefore

$$\frac{w_a}{h_a}\left(\sqrt{\frac{m_b}{h_c}} + \sqrt{\frac{m_c}{h_b}}\right) \leq \sqrt{\frac{R}{2r}} \cdot \sqrt{\frac{2R}{r}} = \frac{R}{r}.$$

1384. In any $\Delta\, ABC$, the following relationship holds :

$$\frac{a^2 r_a^4 h_a}{r_a h_a + m_a a^2} + \frac{b^2 r_b^4 h_b}{r_b h_b + m_b b^2} + \frac{c^2 r_c^4 h_c}{r_c h_c + m_c c^2} \geq \frac{2916 r^5}{\frac{r_a^3 + r_b^3 + r_c^3}{r_a r_b r_c} + 6R}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2r_a^4h_a}{r_ah_a+m_aa^2}+\frac{b^2r_b^4h_b}{r_bh_b+m_bb^2}+\frac{c^2r_c^4h_c}{r_ch_c+m_cc^2}=\sum_{cyc}\frac{r_a^4}{\frac{r_a}{a^2}+\frac{m_a}{h_a}}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc}r_a^2\right)^2}{\sum_{cyc}\frac{r_a}{a^2}+\sum_{cyc}\frac{m_a}{h_a}}\overset{\substack{\text{Chebyshev}\\ \text{and}\\ \text{Panaitopol}}}{\geq}\frac{\left(\sum_{cyc}r_a^2\right)^2}{\frac{1}{3}*(4R+r)*\sum_{cyc}\frac{1}{a^2}+\frac{3R}{2r}}\left(\begin{array}{l}\text{WLOG assuming } a\geq b\geq c\Rightarrow\\ r_a\geq r_b\geq r_c \text{ and } \frac{1}{a^2}\leq\frac{1}{b^2}\leq\frac{1}{c^2}\end{array}\right)$$

$$\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Goldstone}}}{\geq}\frac{\left(\sum_{cyc}r_a^2\right)^2}{\frac{9R}{6}*\frac{4R^2s^2}{16R^2r^2s^2}+\frac{3R}{2r}}=\frac{8r^2\left(\sum_{cyc}r_a^2\right)^2}{3R+12Rr}\overset{?}{\geq}\frac{2916r^5}{\frac{r_a^3+r_b^3+r_c^3}{r_ar_br_c}+6R}$$

$$\Leftrightarrow\boxed{2\left(\sum_{cyc}r_a^2\right)^2*\frac{\sum_{cyc}r_a^3}{rs^2}+12R\left(\sum_{cyc}r_a^2\right)^2\underset{(*)}{\overset{?}{\geq}}2187Rr^3+8748Rr^4}$$

$$\text{Now, }2\left(\sum_{cyc}r_a^2\right)^2*\frac{\sum_{cyc}r_a^3}{rs^2}\overset{A-G}{\geq}6\left(\sum_{cyc}r_ar_b\right)^2=\frac{6}{4}(4s^4)\overset{\text{Gerretsen+Euler}}{\geq}\frac{3}{2}(27Rr)^2$$

$$=2187Rr^2.\frac{R}{2}\overset{\text{Euler}}{\geq}2187Rr^3\therefore\boxed{2\left(\sum_{cyc}r_a^2\right)^2*\frac{\sum_{cyc}r_a^3}{rs^2}\overset{(\bullet)}{\geq}2187Rr^3}$$

$$\text{Again, }12R\left(\sum_{cyc}r_a^2\right)^2\geq12R\left(\sum_{cyc}r_ar_b\right)^2=3R(4s^4)\overset{\text{Gerretsen+Euler}}{\geq}3R(27Rr)^2$$

$$\overset{\text{Euler}}{\geq}3R(54r^2)^2=8748Rr^4\therefore\boxed{12R\left(\sum_{cyc}r_a^2\right)^2\overset{(\bullet\bullet)}{\geq}8748Rr^4}\therefore(\bullet)+(\bullet\bullet)$$

$$\Rightarrow(*)\text{ is true}\therefore\frac{a^2r_a^4h_a}{r_ah_a+m_aa^2}+\frac{b^2r_b^4h_b}{r_bh_b+m_bb^2}+\frac{c^2r_c^4h_c}{r_ch_c+m_cc^2}$$

$$\geq\frac{2916r^5}{\frac{r_a^3+r_b^3+r_c^3}{r_ar_br_c}+6R}\ \forall\,\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS and AM − GM inequalities, we have

$$\sum_{cyc}\frac{a^2{r_a}^4h_a}{r_ah_a+m_aa^2}\geq\frac{\left(\sum_{cyc}\sqrt{{r_a}^4h_a}\right)^2}{\sum_{cyc}\left(\frac{r_ah_a}{a^2}+m_a\right)}\geq\frac{9\sqrt[3]{(r_ar_br_c)^4.(h_ah_bh_c)}}{\sum_{cyc}\frac{r_ah_a}{a^2}+\sum_{cyc}m_a}.$$

Also, we have

$$\bullet\ r_ar_br_c = s^2r \overset{Mitrinovic}{\geq} 27r^2.r = (3r)^3. \quad \bullet\ h_ah_bh_c = \frac{2s^2r^2}{R} \overset{Cosnita-Turtoiu}{\geq} \frac{27Rr.r^2}{R} = (3r)^3.$$

$$\bullet\ m_a + m_b + m_c \overset{Leuenberger}{\leq} 4R + r \overset{Euler}{\leq} 4R + \frac{R}{2} = \frac{9R}{2}.$$

$$\bullet \sum_{cyc} \frac{r_ah_a}{a^2} = \frac{1}{4F^2}\sum_{cyc} r_a{h_a}^3 \overset{h_a \leq \sqrt{s(s-a)}=\sqrt{r_br_c}}{\leq} \frac{r_ar_br_c}{4F^2}\sum_{cyc}\sqrt{r_br_c} \overset{AM-GM}{\leq} \frac{\sum_{cyc} r_a}{4r} = \frac{\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_br_c\right)}{4s^2r}$$

$$\leq \frac{\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} {r_a}^2\right)}{4r_ar_br_c} \overset{Chebyshev}{\leq} \frac{3({r_a}^3 + {r_b}^3 + {r_c}^3)}{4r_ar_br_c}.$$

Using these results, we have

$$\sum_{cyc} \frac{a^2{r_a}^4h_a}{r_ah_a + m_aa^2} \geq \frac{9(3r)^5}{\frac{3({r_a}^3 + {r_b}^3 + {r_c}^3)}{4r_ar_br_c} + \frac{9R}{2}} = \frac{2916r^5}{\frac{{r_a}^3 + {r_b}^3 + {r_c}^3}{r_ar_br_c} + 6R}.$$

Equality holds iff ΔABC is equilateral.

1385. In ΔABC the following relationship holds:

$$\frac{Rw_a}{r\sqrt{r_br_c}} \geq \sqrt{\frac{r_a + r_b + r_c}{h_a + h_b + h_c} \cdot \left(\sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{m_b}{h_c}} + \sqrt{\frac{m_c}{h_b}}\right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : In ΔABC, we have

$$\frac{b}{c} + \frac{c}{b} \leq \frac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{2F} \quad (*)$$

Proof : Since $16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$, then we have

$$(*) \Leftrightarrow (b^2 + c^2)\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)} \leq 2bc\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

$$\overset{squaring}{\Leftrightarrow} (2b^2c^2 + b^4 + c^4)[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)] \leq 4b^2c^2(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow\ -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2$$
$$=-[a^2(b^2+c^2)-(b^4+c^4)]^2\le 0,$$

which is true and the proof of the lemma is complete.

Now, since $\sqrt{a},\sqrt{b},\sqrt{c}$ **can be the sides of triangle with area** F' **such that**

$$F'=\frac{1}{4}.\sqrt{2(ab+bc+ca)-(a^2+b^2+c^2)}=\frac{\sqrt{4r(4R+r)}}{4}=\frac{\sqrt{r(r_a+r_b+r_c)}}{2},$$

then

$$\frac{\sqrt{b}}{\sqrt{c}}+\frac{\sqrt{c}}{\sqrt{b}}\le\frac{\sqrt{ab+bc+ca}}{2F'}=\frac{\sqrt{2R(h_a+h_b+h_c)}}{\sqrt{r(r_a+r_b+r_c)}}=\sqrt{\frac{2R}{r}.\left(\frac{h_a+h_b+h_c}{r_a+r_b+r_c}\right)}.$$

$$\Rightarrow\ \frac{w_a}{\sqrt{r_br_c}}=\frac{2\sqrt{bc}}{b+c}=\frac{2}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}\ge\sqrt{\frac{2r}{R}\left(\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\right)}$$

Now we have

$$n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$$
$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_ar_a,\ \text{ and }\ r_a=s\tan\frac{A}{2}.$$

Then

$$\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}=\sqrt{\frac{n_a{}^2+r_a{}^2}{r_an_a}+2}\overset{n_a\ge h_a}{\le}\sqrt{\frac{s^2+s^2\tan^2\frac{A}{2}-2h_ar_a}{r_ah_a}+2}=\sqrt{\frac{a(s-a)}{2r^2\cos^2\frac{A}{2}}}$$
$$=\sqrt{\frac{abc}{2sr^2}}=\sqrt{\frac{2R}{r}}.$$

and since we have
$\frac{m_b}{h_c}+\frac{m_c}{h_b}\le\frac{R}{r}$ **(see, Bogdan Fuștei, Mohamed Amine Ben Ajiba, *New Triangle Inequalities With Brocard's Angle*, Lemma 12, 6 − 7, www.ssmrmh.ro), then**

$$\sqrt{\frac{m_b}{h_c}}+\sqrt{\frac{m_c}{h_b}}\le\sqrt{2\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)}\le\sqrt{\frac{2R}{r}}.$$

Thereore, using these results, we obtain

$$\frac{Rw_a}{r\sqrt{r_br_c}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\cdot\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\cdot\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{m_b}{h_c}}+\sqrt{\frac{m_c}{h_b}}\right)}.$$

Equality holds iff ΔABC is equilateral.

1386. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left(w_a^2.\sqrt[3]{w_bw_c}\right) \leq p^2.\sqrt[3]{\frac{1}{3}p^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a \geq b \geq c$ and then : $w_a^2 \leq w_b^2 \leq w_c^2$ and $\sqrt[3]{w_bw_c} \geq \sqrt[3]{w_cw_a} \geq \sqrt[3]{w_aw_b}$ $\therefore$ via Chebyshev,

$$\sum_{cyc}\left(w_a^2.\sqrt[3]{w_bw_c}\right) \leq \frac{1}{3}\left(\sum_{cyc}w_a^2\right)\left(\sum_{cyc}\sqrt[3]{w_bw_c}\right)$$

$$\leq \frac{1}{3}\left(\sum_{cyc}p(p-a)\right)\left(\sum_{cyc}\left(\sqrt[3]{w_aw_bw_c}.\frac{1}{\sqrt[3]{w_a}}\right)\right)$$

$$\leq \frac{s^2}{3}*\sqrt[3]{\sqrt{p(p-a)}.\sqrt{p(p-b)}.\sqrt{p(p-c)}}*\sum_{cyc}\frac{1}{\sqrt[3]{h_a}} = \frac{p^2}{3}*\sqrt[3]{\frac{p.rp}{2rp}}*\sum_{cyc}\sqrt[3]{a}$$

$$\overset{Holder}{\leq} \frac{p^2}{3}*\sqrt[3]{\frac{p.}{2}}*\sqrt[3]{9\sum_{cyc}a} = \frac{p^2}{3}*\sqrt[3]{\frac{9p.2p}{2}} = p^2.\sqrt[3]{\frac{1}{3}p^2}$$

$$\therefore \sum_{cyc}\left(w_a^2.\sqrt[3]{w_bw_c}\right) \leq p^2.\sqrt[3]{\frac{1}{3}p^2}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$

1387. In any Δ ABC, the following relationship holds :

$$\frac{w_a^3(m_b^3+r_c^3)}{w_a^3+m_b^3+r_c^3}+\frac{w_b^3(m_c^3+r_a^3)}{w_b^3+m_c^3+r_a^3}+\frac{w_c^3(m_a^3+r_b^3)}{w_c^3+m_a^3+r_b^3} \leq \frac{27(9R^3-64r^3)^2}{32r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$w_a = \frac{2\sqrt{bc}}{b+c}.\sqrt{s(s-a)} \overset{A-G}{\leq} \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-b)(s-c)}} \overset{G-H}{\leq}$$

$$\frac{rs(s-a)(s-b+s-c)}{2(s-a)(s-b)(s-c)}=\frac{rsa(s-a)}{2r^2s}=\frac{a(s-a)}{2r}$$

$$\Rightarrow w_a^3\le s(s-a).\frac{a(s-a)}{2r}\ \textit{and analogs}$$

$$\Rightarrow\sum_{cyc}w_a^3\le\frac{s}{2r}.\sum_{cyc}a(s-a)^2=\frac{s}{2r}.\sum_{cyc}\left(as^2-2sa^2+a^3\right)$$

$$=\frac{s}{2r}.\left(s^2(2s)-4s\left(s^2-4Rr-r^2\right)+2s\left(s^2-6Rr-3r^2\right)\right)$$

$$\Rightarrow\boxed{\sum_{cyc}w_a^3\le(2R-r)s^2}\to(1)$$

$$\text{Now, }m_a^2\overset{?}{\le}\frac{b^3+c^3+abc}{4a}\Leftrightarrow a\left(2b^2+2c^2-a^2\right)\overset{?}{\le}b^3+c^3+abc$$

$$\Leftrightarrow\sum_{cyc}a^3+abc\overset{?}{\ge}2a\left(b^2+c^2\right)$$

$$\Leftrightarrow\sum_{cyc}(y+z)^3+\prod_{cyc}(y+z)\overset{?}{\ge}2(y+z)\left((z+x)^2+(x+y)^2\right)$$

$$\left(\begin{matrix}x=s-a,y=s-b,z=s-c\Rightarrow x+y+z=3s-2s=s\\ \Rightarrow a=y+z,b=z+x,c=x+y;x,y,z>0\end{matrix}\right)$$

$$\Leftrightarrow x^3+y^2z+yz^2\overset{?}{\ge}3xyz\to\text{true via }A-G\ \therefore\ m_a^2\le\frac{b^3+c^3+abc}{4a}$$

$$\Rightarrow m_a^3\overset{Panaitopol}{\le}\frac{b^3+c^3+abc}{4a}.\frac{Rs}{a}=\frac{Rs}{4.16R^2r^2s^2}.b^2c^2\left(b^3+c^3+abc\right)$$

$$\Rightarrow m_a^3\le\frac{1}{64Rr^2s}\left(b^2c^2\left(\sum_{cyc}a^3+abc\right)-a^3b^2c^2\right)\ \textit{and analogs}$$

$$\Rightarrow\sum_{cyc}m_a^3\le\frac{1}{64Rr^2s}\left(\left(\sum_{cyc}a^3+abc\right)\left(\sum_{cyc}b^2c^2\right)-16R^2r^2s^2(2s)\right)$$

$$\overset{Goldstone}{\le}\frac{\left(2s\left(s^2-6Rr-3r^2\right)+4Rrs\right)\left(4R^2s^2\right)-16R^2r^2s^2(2s)}{64Rr^2s}$$

$$=\frac{2s.4R^2s^2\left(s^2-4Rr-7r^2\right)}{64Rr^2s}\Rightarrow\boxed{\sum_{cyc}m_a^3\le\frac{Rs^2\left(s^2-4Rr-7r^2\right)}{8r^2}}\to(2)$$

$$\text{Again, }r_a\le\frac{a^2}{4r}\Rightarrow r_a^3\le s^2\tan^2\frac{A}{2}.\frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{4r}\Rightarrow\sum_{cyc}r_a^3\le\frac{4R^2s^2}{r}\sum_{cyc}\sin^4\frac{A}{2}$$

$$=\frac{4R^2s^2}{r}\left(\left(\sum_{cyc}\sin^2\frac{A}{2}\right)^2-2\sum_{cyc}\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right)$$

$$=\frac{4R^2s^2}{r}\left(\frac{(2R-r)^2}{4R^2}-\frac{2r^2}{16R^2}\sum_{cyc}\frac{bc(s-a)}{r^2s}\right)$$

$$=\frac{4R^2s^2}{r}\left(\frac{(2R-r)^2}{4R^2}-\frac{1}{8R^2}\cdot\frac{s(s^2+4Rr+r^2)-12Rrs}{s}\right)$$

$$\Rightarrow\boxed{\sum_{cyc}r_a^3\le\frac{s^2(8R^2+r^2-s^2)}{2r}}\to(3)$$

Also, $\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3\overset{A-G}{\ge}\sum_{cyc}m_aw_a(m_a+w_a)+3r_ar_br_c\overset{\substack{Lascu+A-G\\ and\\ Tereshin}}{\ge}$

$$\sum_{cyc}\left(s(s-a)\left(\frac{b^2+c^2}{4R}+\frac{bc}{2R}\right)\right)+3rs^2\overset{A-G}{\ge}\sum_{cyc}\frac{s(s-a)(4bc)}{4R}+3rs^2$$

$$=\frac{s.4Rrs}{R}\sum_{cyc}\frac{s-a}{a}+3rs^2=\frac{4rs^2(s^2-8Rr+r^2)}{4Rr}+3rs^2$$

$$\Rightarrow\boxed{\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3\ge\frac{s^2(s^2-5Rr+r^2)}{R}}\to(4)$$

We have : $\frac{w_a^3(m_b^3+r_c^3)}{w_a^3+m_b^3+r_c^3}+\frac{w_b^3(m_c^3+r_a^3)}{w_b^3+m_c^3+r_a^3}+\frac{w_c^3(m_a^3+r_b^3)}{w_c^3+m_a^3+r_b^3}$

$$=\sum_{cyc}\frac{w_a^3(w_a^3+m_b^3+r_c^3-w_a^3)}{w_a^3+m_b^3+r_c^3}=\sum_{cyc}w_a^3-\sum_{cyc}\frac{w_a^6}{w_a^3+m_b^3+r_c^3}\overset{Bergstrom}{\le}$$

$$\sum_{cyc}w_a^3-\frac{\left(\sum_{cyc}w_a^3\right)^2}{\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3}$$

$$=\frac{\left(\sum_{cyc}w_a^3\right)\left(\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3\right)-\left(\sum_{cyc}w_a^3\right)^2}{\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3}$$

$$=\frac{\left(\sum_{cyc}w_a^3\right)\left(\sum_{cyc}m_a^3+\sum_{cyc}r_a^3\right)}{\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3}\overset{via\ (1),(2),(3),(4)}{\le}\frac{(2R-r)s^2.\left(\frac{Rs^2(s^2-4Rr-7r^2)}{8r^2}+\frac{s^2(8R^2+r^2-s^2)}{2r}\right)}{\frac{s^2(s^2-5Rr+r^2)}{R}}$$

$$\overset{Gerretsen}{\le}\frac{R(2R-r)s^2}{11Rr-4r^2}\cdot\frac{R(4R^2+4Rr+3r^2-4Rr-7r^2)+4r(8R^2+r^2-16Rr+5r^2)}{8r^2}$$

$$\overset{?}{\le}\frac{27(9R^3-64r^3)^2}{32r^3}$$

$$\Leftrightarrow 883t^7-384t^6+168t^5-12788t^4+4632t^3+45056t-16384\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)\left(\begin{matrix}883t^5+3148t^4+9228t^3\\+11532t^2+13848t+9264\end{matrix}\right)+26720\right)\overset{?}{\ge}0$$

$$\to\text{true}\because t\overset{Euler}{\ge}2\Rightarrow\frac{w_a^3(m_b^3+r_c^3)}{w_a^3+m_b^3+r_c^3}+\frac{w_b^3(m_c^3+r_a^3)}{w_b^3+m_c^3+r_a^3}+\frac{w_c^3(m_a^3+r_b^3)}{w_c^3+m_a^3+r_b^3}$$

$$\le\frac{27(9R^3-64r^3)^2}{32r^3}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have

$$\sum_{cyc}\frac{w_a^3(m_b^3+r_c^3)}{w_a^3+m_b^3+r_c^3}=\sum_{cyc}\left(w_a^3-\frac{(w_a^3)^2}{w_a^3+m_b^3+r_c^3}\right)$$

$$\le\sum_{cyc}w_a^3-\frac{\left(\sum_{cyc}w_a^3\right)^2}{\sum_{cyc}(w_a^3+m_a^3+r_a^3)}$$

$$=\frac{\sum_{cyc}w_a^3.\left(\sum_{cyc}m_a^3+\sum_{cyc}r_a^3\right)}{\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3}.$$

Also, we have

$$r_a^3+r_b^3+r_c^3=(r_a+r_b+r_c)^3-3(r_a+r_b)(r_b+r_c)(r_a+r_c)$$

$$=(4R+r)^3-3.4Rs^2\overset{Eulzr\,\&\,Mitrinovic}{\le}\left(\frac{9R}{2}\right)^3-12.2r.27r^2=\frac{81(9R^3-64r^3)}{8}.$$

$$w_a^3+w_b^3+w_c^3\le m_a^3+m_b^3+m_c^3$$

$$=(m_a+m_b+m_c)^3-3(m_a+m_b)(m_b+m_c)(m_a+m_c)$$

$$\overset{Leuenberger}{\le}(4R+r)^3-3.8m_am_bm_c\overset{Euler\,\&\,m_a\ge\sqrt{s(s-a)}}{\le}\left(\frac{9R}{2}\right)^3-24s^2r\overset{Mitrinovic}{\le}\frac{81(9R^3-64r^3)}{8}.$$

$$\sum_{cyc}w_a^3+\sum_{cyc}m_a^3+\sum_{cyc}r_a^3\overset{AM-GM}{\ge}3w_aw_bw_c+3m_am_bm_c+3r_ar_br_c$$

$$\ge 6h_ah_bh_c+3s^2r$$

$$=\frac{12s^2}{R}.r^2+3s^2r\overset{Cosnita-Turtoiu\,\&\,Mitrinovic}{\ge}\frac{12.27r}{2}.r^2+3.27r^3=243r^3.$$

Using these results, we have

$$\sum_{cyc}\frac{w_a^3(m_b^3+r_c^3)}{w_a^3+m_b^3+r_c^3}\le\frac{2\left(\frac{81(9R^3-64r^3)}{8}\right)^2}{243r^3}=\frac{27(9R^3-64r^3)^2}{32r^3}.$$

Equality holds if and onl if ΔABC is equilateral.

1388. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\cdot\left(\frac{a}{r_a}\right)^2+\frac{y+z}{x}\cdot\left(\frac{b}{r_b}\right)^2+\frac{z+x}{y}\cdot\left(\frac{c}{r_c}\right)^2\ge 2+8\left(\frac{r_a+r_b+r_c}{a+b+c}\right)^2$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{x+y}{z}\cdot\left(\frac{a}{r_a}\right)^2+\frac{y+z}{x}\cdot\left(\frac{b}{r_b}\right)^2+\frac{z+x}{y}\cdot\left(\frac{c}{r_c}\right)^2=$$

$$=\left(\frac{x+y+z}{z}-1\right)\cdot\left(\frac{a}{r_a}\right)^2+\left(\frac{y+z+x}{x}-1\right)\cdot\left(\frac{b}{r_b}\right)^2+\left(\frac{z+x+y}{y}-1\right)\cdot\left(\frac{c}{r_c}\right)^2=$$

$$=(x+y+z)\sum_{cyc}\frac{\left(\frac{a}{r_a}\right)^2}{z}-\sum_{cyc}\left(\frac{a}{r_a}\right)^2\overset{BERGSTROM}{\geq}$$

$$=(x+y+z)\cdot\frac{\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)^2}{x+y+z}-\sum_{cyc}\left(\frac{a}{r_a}\right)^2=$$

$$=\sum_{cyc}\left(\frac{a}{r_a}\right)^2+2\sum_{cyc}\frac{ab}{r_ar_b}-\sum_{cyc}\left(\frac{a}{r_a}\right)^2=$$

$$=2\sum_{cyc}\frac{ab}{\frac{F}{s-a}\cdot\frac{F}{s-b}}=2\sum_{cyc}\frac{ab}{\frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)}}=2\sum_{cyc}\frac{ab}{s(s-c)}=$$

$$=\frac{2}{s}\sum_{cyc}\frac{ab}{s-c}=\frac{2abc}{s}\sum_{cyc}\frac{1}{c(s-c)}=$$

$$=\frac{2\cdot 4Rrs}{s}\cdot\frac{s^2+(4R+r)^2}{4s^2Rr}=2\left(1+\frac{(4R+r)^2}{s^2}\right)=$$

$$=2\left(1+\frac{(r_a+r_b+r_c)^2}{\frac{(a+b+c)^2}{4}}\right)=2+8\left(\frac{r_a+r_b+r_c}{a+b+c}\right)^2$$

1389. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\cdot\frac{1}{a^2}+\frac{y+z}{x}\cdot\frac{1}{b^2}+\frac{z+x}{y}\cdot\frac{1}{c^2}\geq\frac{1}{Rr}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{x+y}{z}\cdot\frac{1}{a^2}+\frac{y+z}{x}\cdot\frac{1}{b^2}+\frac{z+x}{y}\cdot\frac{1}{c^2}=$$

$$=\left(\frac{x+y+z}{z}-1\right)\cdot\frac{1}{a^2}+\left(\frac{x+y+z}{x}-1\right)\cdot\frac{1}{b^2}+\left(\frac{x+y+z}{y}-1\right)\cdot\frac{1}{c^2}=$$

$$=(x+y+z)\left(\frac{\frac{1}{a^2}}{z}+\frac{\frac{1}{b^2}}{x}+\frac{\frac{1}{c^2}}{y}\right)-\sum_{cyc}\frac{1}{a^2}\overset{BERGSTROM}{\geq}(x+y+z)\cdot\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{x+y+z}-\sum_{cyc}\frac{1}{a^2}=$$

$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2-\sum_{cyc}\frac{1}{a^2}=2\sum_{cyc}\frac{1}{ab}=2\cdot\frac{a+b+c}{abc}=2\cdot\frac{2s}{4RF}=\frac{s}{Rrs}=\frac{1}{Rr}$$

Equality holds for $a=b=c, x=y=z$.

1390. **In any Δ ABC, the following relationship holds :**

$$\frac{(w_a^3+m_b^3)^3}{h_b^3+r_c^3}+\frac{(w_b^3+m_c^3)^3}{h_c^3+r_a^3}+\frac{(w_c^3+m_a^3)^3}{h_a^3+r_b^3}\geq\frac{9\cdot 6^5\cdot r^9}{9R^3-64r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2\overset{?}{\leq}\frac{b^3+c^3+abc}{4a}\Leftrightarrow a(2b^2+2c^2-a^2)\overset{?}{\leq}b^3+c^3+abc$$

$$\Leftrightarrow\sum_{cyc}a^3+abc\overset{?}{\geq}2a(b^2+c^2)$$

$$\Leftrightarrow\sum_{cyc}(y+z)^3+\prod_{cyc}(y+z)\overset{?}{\geq}2(y+z)\left((z+x)^2+(x+y)^2\right)$$

$$\left(\begin{array}{c}x=s-a, y=s-b, z=s-c\Rightarrow x+y+z=3s-2s=s\\ \Rightarrow a=y+z, b=z+x, c=x+y; x,y,z>0\end{array}\right)$$

$$\Leftrightarrow x^3+y^2z+yz^2\overset{?}{\geq}3xyz\rightarrow\text{true via A}-\text{G}\therefore m_a^2\leq\frac{b^3+c^3+abc}{4a}$$

$$\Rightarrow m_a^3\overset{Panaitopol}{\leq}\frac{b^3+c^3+abc}{4a}.\frac{Rs}{a}=\frac{Rs}{4.16R^2r^2s^2}.b^2c^2(b^3+c^3+abc)$$

$$\Rightarrow m_a^3\leq\frac{1}{64Rr^2s}\left(b^2c^2\left(\sum_{cyc}a^3+abc\right)-a^3b^2c^2\right)\text{ and analogs}$$

$$\Rightarrow \sum_{cyc} m_a^3 \le \frac{1}{64Rr^2s}\left(\left(\sum_{cyc} a^3 + abc\right)\left(\sum_{cyc} b^2c^2\right) - 16R^2r^2s^2(2s)\right) \overset{\text{Goldstone}}{\le}$$

$$\frac{\left(2s(s^2-6Rr-3r^2)+4Rrs\right)(4R^2s^2) - 16R^2r^2s^2(2s)}{64Rr^2s} = \frac{2s.4R^2s^2(s^2-4Rr-7r^2)}{64Rr^2s}$$

$$\Rightarrow \boxed{\sum_{cyc} m_a^3 \le \frac{Rs^2(s^2-4Rr-7r^2)}{8r^2}} \rightarrow (1)$$

$$\text{Again}, r_a \le \frac{a^2}{4r} \Rightarrow r_a^3 \le s^2\tan^2\frac{A}{2}.\frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{4r} \Rightarrow \sum_{cyc} r_a^3 \le \frac{4R^2s^2}{r}\sum_{cyc}\sin^4\frac{A}{2}$$

$$= \frac{4R^2s^2}{r}\left(\left(\sum_{cyc}\sin^2\frac{A}{2}\right)^2 - 2\sum_{cyc}\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right)$$

$$= \frac{4R^2s^2}{r}\left(\frac{(2R-r)^2}{4R^2} - \frac{2r^2}{16R^2}\sum_{cyc}\frac{bc(s-a)}{r^2s}\right)$$

$$= \frac{4R^2s^2}{r}\left(\frac{(2R-r)^2}{4R^2} - \frac{1}{8R^2}.\frac{s(s^2+4Rr+r^2)-12Rrs}{s}\right)$$

$$\Rightarrow \boxed{\sum_{cyc} r_a^3 \le \frac{s^2(8R^2+r^2-s^2)}{2r}} \rightarrow (2)$$

$$\text{We have}: \frac{(w_a^3+m_b^3)^3}{h_b^3+r_c^3} + \frac{(w_b^3+m_c^3)^3}{h_c^3+r_a^3} + \frac{(w_c^3+m_a^3)^3}{h_a^3+r_b^3} \overset{\text{Holder}}{\ge}$$

$$\frac{\left(\sum_{cyc} w_a^3 + \sum_{cyc} m_a^3\right)^3}{3\left(\sum_{cyc} h_a^3 + \sum_{cyc} r_a^3\right)} \overset{A-G}{\ge} \frac{\left(6 * \sqrt[6]{\prod_{cyc}(m_a^3 w_a^3)}\right)^3}{3\left(\sum_{cyc} m_a^3 + \sum_{cyc} r_a^3\right)} \overset{\text{via (1) and (2) + Gerretsen}}{\ge}$$

$$\frac{6^3 * \left(\prod_{cyc}(m_a w_a)\right)^{\frac{3}{2}}}{3\left(\frac{Rs^2(4R^2+4Rr+3r^2-4Rr-7r^2)}{8r^2} + \frac{s^2(8R^2+r^2-16Rr+5r^2)}{2r}\right)} \overset{\text{Lascu}}{\ge} \frac{6^3 * \left(\prod_{cyc}(s(s-a))\right)^{\frac{3}{2}} * 2r^2}{3s^2(R^3+8R^2r-17Rr^2+6r^3)}$$

$$= \frac{6^3 * s^6r^3 * 2r^2}{3s^2(R^3+8R^2r-17Rr^2+6r^3)} \overset{\text{Mitrinovic}}{\ge} \frac{6^3 * 729r^4 * 2r^5}{3(R^3+8R^2r-17Rr^2+6r^3)} \overset{?}{\ge} \frac{9 * 6^5 * r^9}{9R^3-64r^3}$$

$$\Leftrightarrow 3(9R^3-64r^3) \overset{?}{\ge} 2(R^3+8R^2r-17Rr^2+6r^3)$$

$$\Leftrightarrow 25t^3 - 16t^2 + 34t - 204 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(25t^2+34t+102) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \frac{(w_a^3+m_b^3)^3}{h_b^3+r_c^3} + \frac{(w_b^3+m_c^3)^3}{h_c^3+r_a^3} + \frac{(w_c^3+m_a^3)^3}{h_a^3+r_b^3}$$

$$\ge \frac{9 * 6^5 * r^9}{9R^3-64r^3}\ \forall\ \Delta\ ABC, '' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1391. In any Δ ABC, the following relationship holds :

$$\frac{h_a w_b(m_b^5+r_c^5)}{m_b^5+h_a w_b+r_c^5}+\frac{h_b w_c(m_c^5+r_a^5)}{m_c^5+h_b w_c+r_a^5}+\frac{h_c w_a(m_a^5+r_b^5)}{m_a^5+h_c w_a+r_b^5}\le\left(\frac{27R}{8r}\right)^2\cdot\frac{81R^5-2560r^5}{54r^3+1}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} m_a^5+\sum_{cyc} r_a^5=\left(\sum_{cyc} m_a\right)^5-5\left(\sum_{cyc} m_a^2+\sum_{cyc} m_a m_b\right)\left(\prod_{cyc}(m_b+m_c)\right)$$

$$+\left(\sum_{cyc} r_a\right)^5-5\left(\sum_{cyc} r_a^2+\sum_{cyc} r_a r_b\right)\left(\prod_{cyc}(r_b+r_c)\right)\overset{\substack{\text{Leuenberger + Euler,}\\ \text{Cesaro,}\\ A-G}}{\le}$$

$$2\left(4R+\frac{R}{2}\right)^5-5\left(6*\sqrt[3]{m_a^2m_b^2m_c^2}\right)(8m_am_bm_c)-5\left(2\sum_{cyc} r_a r_b\right)(8r_ar_br_c)$$

$$\le 2\left(\frac{9R}{2}\right)^5-5\left(6*\sqrt[3]{h_a^2h_b^2h_c^2}\right)(8h_ah_bh_c)-5\left(2\sum_{cyc} r_a r_b\right)(8r_ar_br_c)$$

$$=2\left(\frac{9R}{2}\right)^5-5\left(6*\sqrt[3]{\frac{4r^4s^4}{R^2}}\right)\left(8.\frac{2r^2s^2}{R}\right)-5.2s^2.8rs^2$$

$$\le 2\left(\frac{9R}{2}\right)^5-5\left(6*\sqrt[3]{\frac{r^4.729R^2r^2}{R^2}}\right)\left(8.\frac{r^2.27Rr}{R}\right)-5.2.8r.729r^4$$

$$\left(\because 2s^2\overset{\text{Gerretsen}}{\ge}27Rr+5r(R-2r)\overset{\text{Euler}}{\ge}27Rr \text{ and } s^2\overset{\text{Mitrinovic}}{\ge}27r^2\right)$$

$$=\frac{729*81R^5}{16}-2*5*2*8*729r^5\Rightarrow\sum_{cyc} m_a^5+\sum_{cyc} r_a^5\overset{(\bullet)}{\le}\frac{729(81R^5-2560r^5)}{16}$$

$$\text{and }\sum_{cyc} h_a w_b\overset{\text{CBS}}{\le}\sqrt{\sum_{cyc} h_a^2}*\sqrt{\sum_{cyc} w_a^2}\le\sum_{cyc} w_a^2\le\sum_{cyc} s(s-a)\overset{\text{Mitrinovic}}{\le}\frac{27R^2}{4}$$

$$\Rightarrow\sum_{cyc} h_a w_b\overset{(\bullet\bullet)}{\le}\frac{27R^2}{4}\text{ and we have :}$$

$$\frac{h_a w_b(m_b^5+r_c^5)}{m_b^5+h_a w_b+r_c^5}+\frac{h_b w_c(m_c^5+r_a^5)}{m_c^5+h_b w_c+r_a^5}+\frac{h_c w_a(m_a^5+r_b^5)}{m_a^5+h_c w_a+r_b^5}$$

$$=\sum_{cyc}\frac{h_a w_b(m_b^5+r_c^5+h_a w_b-h_a w_b)}{m_b^5+h_a w_b+r_c^5}=\sum_{cyc}h_a w_b-\sum_{cyc}\frac{h_a^2 w_b^2}{m_b^5+h_a w_b+r_c^5}\overset{Bergstrom}{\leq}$$

$$\sum_{cyc}h_a w_b-\frac{\left(\sum_{cyc}h_a w_b\right)^2}{\sum_{cyc}m_a^5+\sum_{cyc}r_a^5+\sum_{cyc}h_a w_b}=\frac{\left(\sum_{cyc}m_a^5+\sum_{cyc}r_a^5\right)\left(\sum_{cyc}h_a w_b\right)}{\sum_{cyc}m_a^5+\sum_{cyc}r_a^5+\sum_{cyc}h_a w_b}$$

$$\overset{via\ (\bullet),(\bullet\bullet)}{\leq}\frac{729(81R^5-2560r^5)}{16}.\frac{27R^2}{4}.\frac{1}{\sum_{cyc}m_a^5+\sum_{cyc}r_a^5+\sum_{cyc}h_a w_b}$$

$$\overset{?}{\leq}\left(\frac{27R}{8r}\right)^2.\frac{81R^5-2560r^5}{54r^3+1}\Leftrightarrow\sum_{cyc}m_a^5+\sum_{cyc}r_a^5+\sum_{cyc}h_a w_b\underset{(*)}{\overset{?}{\geq}}1458r^5+27r^2$$

$$\text{Now, }\sum_{cyc}m_a^5+\sum_{cyc}r_a^5+\sum_{cyc}h_a w_b\overset{A-G}{\geq}3(m_a m_b m_c)^{\frac{5}{3}}+3(r_a r_b r_c)^{\frac{5}{3}}+3\sqrt[3]{h_a^2 h_b^2 h_c^2}$$

$$\geq 3(h_a h_b h_c)^{\frac{5}{3}}+3(rs^2)^{\frac{5}{3}}+3*\sqrt[3]{h_a^2 h_b^2 h_c^2}=3*\left(\frac{2r^2s^2}{R}\right)^{\frac{5}{3}}+3(rs^2)^{\frac{5}{3}}+3*\sqrt[3]{\frac{4r^4s^4}{R^2}}$$

$$\overset{\substack{Mitrinovic\\ and\\ 2s^2\geq 27Rr}}{\geq}3*\left(\frac{r^2.27Rr}{R}\right)^{\frac{5}{3}}+3*(r.27r^2)^{\frac{5}{3}}+3*\sqrt[3]{\frac{r^4.729R^2r^2}{R^2}}=1458r^5+27r^2$$

$$\Rightarrow(*)\text{ is true}\therefore\frac{h_a w_b(m_b^5+r_c^5)}{m_b^5+h_a w_b+r_c^5}+\frac{h_b w_c(m_c^5+r_a^5)}{m_c^5+h_b w_c+r_a^5}+\frac{h_c w_a(m_a^5+r_b^5)}{m_a^5+h_c w_a+r_b^5}$$

$$\leq\left(\frac{27R}{8r}\right)^2.\frac{81R^5-2560r^5}{54r^3+1}\ \forall\Delta ABC,''='' \text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1392. In any Δ ABC, the following relationship holds :

$$\frac{m_a^5}{(h_b w_c)^3}+\frac{m_b^5}{(h_c w_a)^3}+\frac{m_c^5}{(h_a w_b)^3}\geq\frac{64r^5}{R^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^5}{(h_b w_c)^3}+\frac{m_b^5}{(h_c w_a)^3}+\frac{m_c^5}{(h_a w_b)^3}=\sum_{cyc}\frac{\left(\frac{m_a}{h_b w_c}\right)^3}{\left(\frac{1}{m_a}\right)^2}\overset{Radon}{\geq}\frac{\left(\sum_{cyc}\frac{m_a}{h_b w_c}\right)^3}{\left(\sum_{cyc}\frac{1}{m_a}\right)^2}$$

$$\geq\frac{\left(\sum_{cyc}\frac{m_a}{w_b w_c}\right)^3}{\left(\sum_{cyc}\frac{1}{h_a}\right)^2}\overset{\substack{m_a\geq\sqrt{s(s-a)}\text{ and analogs via Lascu + A-G}\\ and\\ w_a\leq\sqrt{s(s-a)}\text{ and analogs}}}{\geq}\frac{\left(\sum_{cyc}\frac{\sqrt{s(s-a)}*\sqrt{s(s-a)}}{\sqrt{s(s-b)}*\sqrt{s(s-c)}*\sqrt{s(s-a)}}\right)^3}{\left(\frac{1}{r}\right)^2}$$

$$= r^2 * \left(\frac{s}{s.rs} * \sum_{cyc}(s-a)\right)^3 = \frac{r^2}{r^3} = \frac{1}{r} = \frac{r^5}{r^6} \overset{\text{Euler}}{\geq} \frac{r^5}{\left(\frac{R}{2}\right)^6} = \frac{64r^5}{R^6},$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1393. **In ΔABC holds:**

$$\sqrt{2}a\cos\frac{B}{2}\cos\frac{C}{2} = s \Rightarrow \sec(2B) + \tan(2B) = \frac{c+b}{c-b}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\sqrt{2}a\sqrt{\frac{s(s-b)}{ab}}\cdot\sqrt{\frac{s(s-c)}{ac}} = s \Rightarrow \sqrt{2}\sqrt{\frac{(s-b)(s-c)}{bc}} = 1$$

$$\Rightarrow \sin\frac{A}{2} = 1 \Rightarrow \frac{A}{2} = \frac{\pi}{4} \Rightarrow A = \frac{\pi}{2}$$

Now, $\sec(2B) + \tan(2B) = \frac{1+\sin(2B)}{\cos(2B)} = \frac{(\sin B+\cos B)^2}{\cos^2 B-\sin^2 B} = \frac{\cos B+\sin B}{\cos B-\sin B} = \frac{\sin C+\sin B}{\sin C-\sin B}$

$$\left[\because B + C = \frac{\pi}{2}\right]$$

$= \frac{c+b}{c-b}$ [using law of sines]

Solution 2 by Cosqun Memmedoff-Azerbaijan

$$\sqrt{2}\,a\cos\frac{B}{2}\cos\frac{C}{2} = s,\ s = \frac{a+b+c}{2}$$

$$s^2 = 2a^2\cos^2\frac{B}{2}\cos^2\frac{C}{2}$$

$$s^2 = 2a^2\left(\frac{1+\cos B}{2}\right)\left(\frac{1+\cos C}{2}\right), \cos B = \frac{a^2+c^2-b^2}{2ac}, \cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$s^2 = 2a^2\left(\frac{1}{2} + \frac{a^2+c^2-b^2}{4ac}\right)\left(\frac{1}{2} + \frac{a^2+b^2-c^2}{4ab}\right)$$

$$s^2 = 2a^2\left(\frac{(a+c)^2-b^2}{4ac}\right)\left(\frac{(a+b)^2-c^2}{4ab}\right)$$

$$s^2=\frac{(a+b+c)^2(a+c-b)(a+b-c)}{8bc}$$

$$\frac{(a+b+c)^2}{4}=\frac{(a+b+c)^2(a+c-b)(a+b-c)}{8bc}$$

$$(a+c-b)(a+b-c)=2bc$$

$$a^2+ab-ac+ca+cb-c^2-ab-b^2+bc=2bc$$

$$a^2-c^2-b^2=0\Leftrightarrow a^2=b^2+c^2\Rightarrow \cos B=\frac{c}{a},\sin B=\frac{b}{a}$$

$$\sec(2B)+\tan(2B)=\frac{1}{\cos(2B)}+\frac{\sin(2B)}{\cos(2B)}=\frac{\sin(2B)+1}{\cos 2B}=$$

$$=\frac{\cos B+\sin B}{\cos B-\sin B}=\frac{\frac{c}{a}+\frac{b}{a}}{\frac{c}{a}-\frac{b}{a}}=\frac{c+b}{c-b}$$

Solution 3 by Tapas Das-India

$$\sqrt{2}a\cos\frac{B}{2}\cos\frac{C}{2}=s$$

$$\sqrt{2}\cdot 2R\cdot\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\cos\frac{B}{2}\cdot\cos\frac{C}{2}=s$$

$$2\sqrt{2}\times 2\times R\cdot\sin\frac{A}{2}\cdot\frac{s}{4R}=s$$

$$\sqrt{2}\sin\frac{A}{2}=1\therefore \sin\frac{A}{2}=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$$

$$A=\frac{\pi}{2}$$

$$\therefore \cos B=\frac{c}{a}$$

$$\sin B=\frac{b}{a}$$

$$\sec 2B+\tan 2B=\frac{1+\sin 2B}{\cos 2B}=\frac{(\sin B+\cos B)^2}{(\cos B+\sin B)(\cos B-\sin B)}$$

$$=\frac{\cos B+\sin B}{\cos B-\sin B}=\frac{\frac{c}{a}+\frac{b}{a}}{\frac{c}{a}-\frac{b}{a}}=\frac{c+b}{c-b}$$

1394. In ΔABC holds:

$$\sum_{cyc} a^3 = 6sR^2 \Rightarrow \prod_{cyc}(b^2+c^2-bc-a^2) = 0$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das-India

$$(b^2+c^2-bc-a^2) = b^2+c^2-a^2-bc = 2bc\cos A - bc = bc(2\cos A - 1)$$

(analog)

$$\therefore \prod_{cyc} bc(2\cos A - 1)$$

$$= ab\cdot bc\cdot ca(2\cos A-1)(2\cos B-1)(2\cos C-1)$$

$$= a^2b^2c^2\{-(1-2\cos A)(1-2\cos B)(1-2\cos C)\}$$

$$= -a^2b^2c^2\begin{bmatrix} 1-2(\cos A+\cos B+\cos C)+ \\ +4(\cos A\cos B+\cos B\cos C+\cos C\cos A)- \\ -8\cos A\cos B\cos C \end{bmatrix}$$

$$= -a^2b^2c^2\left[1-2\left(1+\frac{r}{R}\right)+4\frac{s^2+r^2-4R^2}{4R^2}-8\frac{s^2-(2R+r)^2}{4R^2}\right]$$

$$= -\frac{a^2b^2c^2}{R^2}[R^2-2R^2-2Rr+s^2+r^2-4R^2-2s^2+2(2R+r)^2]$$

$$= -\frac{a^2b^2c^2}{R^2}[-R^2-2Rr+s^2+r^2-4R^2-2s^2+8R^2+8Rr+2r^2]$$

$$= -\frac{a^2b^2c^2}{R^2}[3R^2+6Rr-s^2+3r^2] = 0$$

Note: $\sum a^3 = 6sR^2,\ \therefore 2s(s^2-3r^2-6Rr) = 6sR^2,\ \therefore 3R^2+6Rr+3r^2-s^2 = 0$

Using the following relationships:

$$\cos A+\cos B+\cos C = 1+\frac{r}{R},\qquad \sum \cos A\cos B = \frac{s^2+r^2-4R^2}{4R^2}$$

$$\prod \cos A = \frac{s^2-(2R+r)^2}{4R^2}$$

1395. In ΔABC the following relationship holds:

$$tan\frac{A}{4}+tan\frac{B}{4}+tan\frac{C}{4} = \frac{AI+BI+CI-s}{r}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

Denote:

$$\tan\frac{A}{4}=x,\qquad \tan\frac{A}{2}=\frac{2\tan\frac{A}{4}}{1-\tan^2\frac{A}{4}}=\frac{2x}{1-x^2}$$

$$(1-x^2)\tan\frac{A}{2}-2x=0,\qquad x^2\tan\frac{A}{2}+2x-\tan\frac{A}{2}=0$$

$$\Delta=4+4\tan^2\frac{A}{2}=\frac{4}{\cos^2\frac{A}{2}},\qquad \tan\frac{A}{4}=x=\frac{-2+\frac{2}{\cos\frac{A}{2}}}{2\tan\frac{A}{2}}=\frac{\frac{1}{\cos\frac{A}{2}}-1}{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}=\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}$$

$$\tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}=\sum_{cyc}\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}=\sum_{cyc}\frac{1}{\sin\frac{A}{2}}-\sum_{cyc}\cot\frac{A}{2}=$$

$$=\sum_{cyc}\frac{1}{\frac{r}{AI}}-\frac{s}{r}=\frac{AI+BI+CI-s}{r}$$

1396. In $\Delta ABC, I_a, I_b, I_c$ −excenters, the following relationship holds:

$$\frac{I_bI_c}{w_a}+\frac{I_cI_a}{w_b}+\frac{I_aI_b}{w_c}=\frac{2R}{F}\cdot(h_a+h_b+h_c)$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{I_bI_c}{w_a}=\sum_{cyc}\frac{4R\cos\frac{A}{2}}{\frac{2bc}{b+c}\cos\frac{A}{2}}=2R\sum_{cyc}\frac{b+c}{bc}=2R\sum_{cyc}\left(\frac{1}{b}+\frac{1}{c}\right)=$$

$$=4R\sum_{cyc}\frac{1}{a}=\frac{2R}{F}\sum_{cyc}\frac{2F}{a}=\frac{2R}{F}\cdot(h_a+h_b+h_c)$$

1397.

In any bicentric quadrilateral ABCD with sides → a, b, c, d, the following

relationship holds :

$$\frac{2s+a}{4s-a}+\frac{2s+b}{4s-b}+\frac{2s+c}{4s-c}+\frac{2s+d}{4s-d}\ge\frac{40}{7}\left(\frac{r}{R}\right)^2$$

Proposed by Emil C. Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Brahmagupta and Parameshvara, } 16F^2R^2$$
$$=(ac+bd)(ab+cd)(ad+bc)$$
$$\Rightarrow x\left(bd\left((a+c)^2-2ac\right)+ac\left((b+d)^2-2bd\right)\right)=16R^2r^2s^2\ (x=ac+bd)$$
$$\Rightarrow x\left(bd(s^2-2ac)+ac(s^2-2bd)\right)=16R^2r^2s^2\Rightarrow x(s^2x-4r^2s^2)=16R^2r^2s^2$$
$$\Rightarrow x^2-4x.r^2-16R^2r^2=0\Rightarrow x=\frac{4r^2\pm\sqrt{64R^2r^2+16r^4}}{2}$$
$$\Rightarrow ac+bd=2r^2+2r*\sqrt{4R^2+r^2}\to(1)$$
$$\therefore\sum_{cyc}a^2=4s^2-2\left(ac+bd+(ab+bc)+(ad+cd)\right)\overset{\text{via }(1)}{=}$$
$$4s^2-4r^2-4r*\sqrt{4R^2+r^2}-2(bs+ds)=4s^2-4r^2-4r*\sqrt{4R^2+r^2}-2s^2$$
$$\therefore\sum_{cyc}a^2=2s^2-4r^2-4r*\sqrt{4R^2+r^2}\to(2)$$
$$\frac{2s+a}{4s-a}+\frac{2s+b}{4s-b}+\frac{2s+c}{4s-c}+\frac{2s+d}{4s-d}\ge\frac{40}{7}\left(\frac{r}{R}\right)^2\Leftrightarrow\sum_{cyc}\frac{4s-a+3a}{4s-a}\ge\frac{80}{7}\left(\frac{r}{R}\right)^2$$
$$\Leftrightarrow 4+3\sum_{cyc}\frac{a}{4s-a}\overset{(*)}{\ge}\frac{80}{7}\left(\frac{r}{R}\right)^2$$
$$\text{Again, }4+3\sum_{cyc}\frac{a}{4s-a}=4+3\sum_{cyc}\frac{a^2}{4sa-a^2}\overset{\text{Bergstrom}}{\ge}4+\frac{12s^2}{4s(2s)-\sum_{cyc}a^2}$$
$$\overset{?}{\ge}\frac{80}{7}\left(\frac{r}{R}\right)^2\Leftrightarrow(77R^2-160r^2)s^2\overset{?}{\ge}\left(\sum_{cyc}a^2\right)(7R^2-20r^2)$$
$$\overset{\text{via }(2)}{\Leftrightarrow}(77R^2-160r^2)s^2\overset{?}{\ge}(7R^2-20r^2)\left(2s^2-4r^2-4r*\sqrt{4R^2+r^2}\right)$$
$$\Leftrightarrow(63R^2-120r^2)s^2+(7R^2-20r^2)\left(4r^2+4r*\sqrt{4R^2+r^2}\right)\underset{(**)}{\overset{?}{\ge}}0$$
$$\because 63R^2-120r^2=63(R^2-2r^2)+6r^2\overset{\text{L.Fejes Toth,1948}}{\ge}6r^2>0$$
$$\therefore\text{ via Blundon - Eddy, LHS of }(**)\ge(63R^2-120r^2)*8r*\left(\sqrt{4R^2+r^2}-r\right)$$
$$+(7R^2-20r^2)\left(4r^2+4r*\sqrt{4R^2+r^2}\right)\overset{?}{\ge}0$$
$$\Leftrightarrow\sqrt{4R^2+r^2}*(133R^2-260r^2)\underset{(***)}{\overset{?}{\ge}}r(119R^2-220r^2)$$
$$\text{Once again, via L. Fejes Toth, LHS of }(***)\ge$$

$$3r(133R^2-260r^2)\overset{?}{\geq} r(119R^2-220r^2)\Leftrightarrow 280R^2\overset{?}{\geq}560r^2$$
$$\to \text{true, via } L.\text{Fejes Toth} \Rightarrow (***)\Rightarrow(**)\Rightarrow(*) \text{ is true}$$
$$\therefore \frac{2s+a}{4s-a}+\frac{2s+b}{4s-b}+\frac{2s+c}{4s-c}+\frac{2s+d}{4s-d}\geq\frac{40}{7}\left(\frac{r}{R}\right)^2$$
$$\forall \text{ bicentric quadrilateral } ABCD \text{ (QED)}$$

1398.

In any bicentric quadrilateral $ABCD$ with $AB=a, BC=b, CD=c, DA=d, AC=e,$ $BD=f, m=\min(e,f)$ and $\frac{R}{r}=\alpha\geq\sqrt{2}$, the following relationship holds:

$$a^2+b^2+c^2+d^2\geq 4\sqrt[3]{\frac{\alpha^2}{m^2}}.F^{\frac{4}{3}}$$

Proposed by Emil. C. Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Brahmagupta and Parameshvara, } 16F^2R^2$$
$$=(ac+bd)(ab+cd)(ad+bc)\Rightarrow x\left(bd\left((a+c)^2-2ac\right)+ac\left((b+d)^2-2bd\right)\right)$$
$$=16R^2r^2s^2\ (x=ac+bd)\Rightarrow x\left(bd(s^2-2ac)+ac(s^2-2bd)\right)=16R^2r^2s^2$$
$$\Rightarrow x(s^2x-4r^2s^2)=16R^2r^2s^2\Rightarrow x^2-4x.r^2-16R^2r^2=0$$
$$\Rightarrow x=\frac{4r^2\pm\sqrt{64R^2r^2+16r^4}}{2}\Rightarrow ac+bd=2r^2+2r*\sqrt{4R^2+r^2}\to(1)$$
$$\therefore\sum_{cyc}a^2=4s^2-2\left(ac+bd+(ab+bc)+(ad+cd)\right)\overset{\text{via }(1)}{=}$$
$$4s^2-4r^2-4r*\sqrt{4R^2+r^2}-2(bs+ds)=4s^2-4r^2-4r*\sqrt{4R^2+r^2}-2s^2$$
$$\therefore\sum_{cyc}a^2=2s^2-4r^2-4r*\sqrt{4R^2+r^2}\to(2)$$
$$\text{Via Ptolemy's first and second theorems: } ef=ac+bd \text{ and } \frac{e}{f}=\frac{ad+bc}{ab+cd}$$
$$\Rightarrow e^2=\frac{(ac+bd)(ab+cd)(ad+bc)}{(ab+cd)^2}\overset{\text{Brahmagupta and Parameshvara}}{=}\frac{16R^2r^2s^2}{(ab+cd)^2}$$
$$\overset{CBS}{\geq}\frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)} \text{ and similarly, } f^2=\frac{(ac+bd)(ab+cd)(ad+bc)}{(ad+bc)^2}$$
$$\overset{CBS}{\geq}\frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)}\therefore e^2,f^2\geq\frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)}\therefore m^2\geq\frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)}$$
$$=\frac{16R^2r^2s^2}{(s^2-2ac)(s^2-2bd)}=\frac{16R^2r^2s^2}{s^4-2s^2(ac+bd)+4abcd}\overset{\text{via }(1)}{=}$$

$$\frac{16R^2r^2s^2}{s^4-2s^2\left(2r^2+2r*\sqrt{4R^2+r^2}\right)+4r^2s^2}\Rightarrow \boxed{m^2\overset{(*)}{\geq}\frac{16R^2r^2}{s^2-4r*\sqrt{4R^2+r^2}}}$$

$$\text{Now, }(a^2+b^2+c^2+d^2)^3\geq\frac{1}{4}\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}a\right)^2\overset{via\,(2)}{=}$$

$$\frac{4s^2}{4}\left((2s^2-4r^2)^2+16r^2(4R^2+r^2)-8(2s^2-4r^2)r*\sqrt{4R^2+r^2}\right)$$

$$\Rightarrow m^2(a^2+b^2+c^2+d^2)^3\geq$$

$$\begin{pmatrix}(2s^2-4r^2)^2+16r^2(4R^2+r^2)\\-8(2s^2-4r^2)r*\sqrt{4R^2+r^2}\end{pmatrix}\left(\frac{16R^2r^2}{s^2-4r*\sqrt{4R^2+r^2}}\right)\overset{?}{\geq}64\alpha^2F^4$$

$$=64R^2r^2s^4\left(\because\frac{R}{r}=\alpha\text{ and }F=rs\right)$$

$$\Leftrightarrow(2s^2-4r^2)^2+16r^2(4R^2+r^2)-8(2s^2-4r^2)r*\sqrt{4R^2+r^2}$$

$$\overset{?}{\geq}4s^2\left(s^2-4r*\sqrt{4R^2+r^2}\right)$$

$$\Leftrightarrow16r^2(4R^2+2r^2-s^2)+(16rs^2-16rs^2+32r^3)*\sqrt{4R^2+r^2}\overset{?}{\geq}0$$

$$\Leftrightarrow s^2\overset{?}{\leq}4R^2+2r^2+2r*\sqrt{4R^2+r^2}=\left(\sqrt{4R^2+r^2}+r\right)^2\Leftrightarrow s\overset{?}{\leq}\sqrt{4R^2+r^2}+r$$

$$\rightarrow\text{true via Blundon}-\text{Eddy}\therefore m^2(a^2+b^2+c^2+d^2)^3\geq64\alpha^2F^4$$

$$\Rightarrow a^2+b^2+c^2+d^2\geq4\sqrt[3]{\frac{\alpha^2}{m^2}}.F^{\frac{4}{3}}\ \forall\text{ bicentric quadrilateral }ABCD\text{ (QED)}$$

1399. In any ΔABC, the following relationship holds :

$$\frac{w_a^5m_b^5}{h_b^2r_c^2}+\frac{w_b^5m_c^5}{h_c^2r_a^2}+\frac{w_c^5m_a^5}{h_a^2r_b^2}\geq\frac{27\cdot6^4\cdot r^{10}}{R^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_c}{h_a}=\sum_{cyc}\left(\frac{rs}{s-c}\cdot\frac{a}{2rs}\right)=\frac{1}{2r^2s}\sum_{cyc}a(s-a)(s-b)$$

$$\overset{A-G}{\leq}\frac{1}{8r^2s}\sum_{cyc}a(2s-a-b)^2=\frac{1}{8r^2s}\sum_{cyc}ac^2\overset{A-G}{\leq}\frac{1}{8r^2s}\sum_{cyc}a^3=\frac{2s(s^2-6Rr-3r^2)}{8r^2s}$$

$$\therefore\sum_{cyc}\frac{r_c}{h_a}\overset{(i)}{\leq}\frac{s^2-6Rr-3r^2}{4r^2}$$

$$\sum_{cyc}w_aw_b\geq\sum_{cyc}h_ah_b=\sum_{cyc}\frac{bc.ca}{4R^2}=\frac{rs}{R}\sum_{cyc}a=\frac{2rs^2}{R}\Rightarrow\sum_{cyc}w_aw_b\overset{(ii)}{\geq}\frac{2rs^2}{R}$$

$$\text{Now, } \frac{w_a^5m_b^5}{h_b^2r_c^2}+\frac{w_b^5m_c^5}{h_c^2r_a^2}+\frac{w_c^5m_a^5}{h_a^2r_b^2}=\sum_{cyc}\frac{(w_am_b)^3}{\left(\frac{h_br_c}{w_am_b}\right)^2}\ge\sum_{cyc}\frac{(w_aw_b)^3}{\left(\frac{r_c}{h_a}\right)^2}\overset{\text{Radon}}{\ge}\frac{\left(\sum_{cyc}w_aw_b\right)^3}{\left(\sum_{cyc}\frac{r_c}{h_a}\right)^2}$$

$$\overset{\text{via (i),(ii)}}{\ge}\frac{8r^3s^6*16r^4}{R^3(s^2-6Rr-3r^2)^2}\overset{\text{Gerretsen}}{\ge}\frac{4r^3*16r^4s^4*(27Rr+5r(R-2r))}{R^3(s^2-6Rr-3r^2)^2}$$

$$\overset{\text{Euler}}{\ge}\frac{4r^3*16r^4s^4*27Rr}{R^3(s^2-6Rr-3r^2)^2}\overset{?}{\ge}\frac{27*6^4*r^{10}}{R^4}=\frac{27*81*16*r^{10}}{R^4}$$

$$\Leftrightarrow 2Rs^2\overset{?}{\ge}9r(s^2-6Rr-3r^2)\Leftrightarrow(2R-9r)s^2+9r(6Rr+3r^2)\underset{(*)}{\overset{?}{\ge}}0$$

$\boxed{Case\ 1}$ $2R-9r\ge 0$ *and then* : LHS of $(*)\ge 9r(6Rr+3r^2)>0$

$\Rightarrow (*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $2R-9r<0$ *and then* : LHS of $(*)=-(9r-2R)s^2+9r(6Rr+3r^2)$

$$\overset{\text{Gerretsen}}{\ge}-(9r-2R)(4R^2+4Rr+3r^2)+9r(6Rr+3r^2)\overset{?}{\ge}0$$

$$\Leftrightarrow 4R(2R^2-7Rr+6r^2)\overset{?}{\ge}0\Leftrightarrow 4R(2R-3r)(R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$\Rightarrow (*)$ is true $\therefore$ combining cases 1 and 2, $(*)$ is true $\forall\,\Delta\,ABC$

$$\therefore\frac{w_a^5m_b^5}{h_b^2r_c^2}+\frac{w_b^5m_c^5}{h_c^2r_a^2}+\frac{w_c^5m_a^5}{h_a^2r_b^2}\ge\frac{27*6^4*r^{10}}{R^4}$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)

1400. In any $\Delta\,ABC$, the following relationship holds :

$$\sum_{cyc}\frac{1}{4\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\le\frac{4R+r}{2s}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{s}{4r_a+r_b+r_c}=\sum_{cyc}\frac{s}{4R+r+3r_a}$$

$$=s*\frac{\sum_{cyc}(4R+r+3r_b)(4R+r+3r_c)}{(4R+r+3r_a)(4R+r+3r_b)(4R+r+3r_c)}$$

$$=s*\frac{3(4R+r)^2+6(4R+r)\left(\sum_{cyc}r_a\right)+9\sum_{cyc}r_br_c}{(4R+r)^3+9(4R+r)\left(\sum_{cyc}r_br_c\right)+3(4R+r)^2\left(\sum_{cyc}r_a\right)+27rs^2}$$

$$=s*\frac{3(4R+r)^2+6(4R+r)(4R+r)+9s^2}{(4R+r)^3+9(4R+r)s^2+3(4R+r)^2(4R+r)+27rs^2}\overset{?}{\le}\frac{4R+r}{2s}$$

$$\Leftrightarrow 2(4R+r)^4\underset{(*)}{\overset{?}{\ge}}(72R^2-18Rr-9r^2)s^2+9s^4$$

$$\text{Now, RHS of }(*)\overset{\text{Gerretsen}}{\le}(72R^2-18Rr-9r^2)s^2+9(4R^2+4Rr+3r^2)s^2$$

$$\overset{?}{\le}2(4R+r)^4\Leftrightarrow(4R+r)^4\underset{(**)}{\overset{?}{\ge}}(54R^2+9Rr+9r^2)s^2$$

$$\text{Again, RHS of } (**) \overset{\text{Gerretsen}}{\leq} (54R^2+9Rr+9r^2)(4R^2+4Rr+3r^2) \overset{?}{\leq} (4R+r)^4$$

$$\Leftrightarrow 40t^4+4t^3-138t^2-47t-26 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(40t^3+84t^2+30t+13) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore (**) \Rightarrow (*) \text{ is true} \therefore$$

$$\sum_{cyc} \frac{1}{4\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}} \leq \frac{4R+r}{2s}\ \forall\, \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru