ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)^{n+\frac{3}{2}}}{n\left(n+\frac{3}{2}\right)\,{}^{2n}C_n}$$

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$$I=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)^{n+\frac{3}{2}}}{n\left(n+\frac{3}{2}\right)\,{}^{2n}C_n}=\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\frac{1}{3^{n+1}\,n(2n+3)\,{}^{2n}C_n}=$$

$$=\frac{2}{\sqrt{3}}\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}\,n!\,n!}{3^{n+1}\,n\,(2n)!}dx=\frac{2}{\sqrt{3}}\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}\,\Gamma(n)\Gamma(n+1)}{3^{n+1}\,\Gamma(2n+1)}dx=$$

$$=\frac{2}{\sqrt{3}}\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}}{3^{n+1}}B(n,n+1)dx=\frac{2}{\sqrt{3}}\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}}{3^{n+1}}(1-t)^{n-1}t^n dtdx=$$

$$=\frac{2}{\sqrt{3}}\int_0^1\int_0^1\frac{\frac{x^4t}{3^2}}{1-\frac{x^2}{3}t(1-t)}dtdx=\frac{2}{3\sqrt{3}}\int_0^1\int_0^1\frac{x^4t}{3-x^2t(1-t)}dtdx=$$

$$\int_0^1\frac{t}{3-x^2t(1-t)}dt\underset{t\to 1-t}{=}\int_0^1\frac{1-t}{3-x^2t(1-t)}dt=\frac{1}{2}\int_0^1\frac{1}{3-x^2t(1-t)}dt=$$

$$=\frac{1}{2}\int_0^1\frac{1}{3-\frac{x^2}{4}+\left(xt-\frac{x}{2}\right)^2}dt=\frac{1}{2x\sqrt{3-\frac{x^2}{4}}}\left[\tan^{-1}\left(\frac{xt-\frac{x}{2}}{\sqrt{3-\frac{x^2}{4}}}\right)\right]_{t=0}^{t=1}=$$

$$=\frac{2}{x\sqrt{12-x^2}}\tan^{-1}\left(\frac{x}{\sqrt{12-x^2}}\right)=\frac{2}{x\sqrt{12-x^2}}\sin^{-1}\left(\frac{x}{\sqrt{12}}\right)$$

Therefore,

$$I=\frac{2}{3\sqrt{3}}\int_0^1 \frac{2x^4}{x\sqrt{12-x^2}}\sin^{-1}\left(\frac{x}{\sqrt{12}}\right)dx=\frac{4}{3\sqrt{3}}\int_0^1 \frac{x^3}{\sqrt{12-x^2}}\sin^{-1}\left(\frac{x}{\sqrt{12}}\right)dx$$

$$=32\int_0^{\phi}\theta\sin^3\theta d\theta$$

$$\text{by substituting }\ \theta=\sin^{-1}\left(\frac{x}{\sqrt{12}}\right)\text{ or } x=\sqrt{12}\sin\theta$$

$$=8\int_0^{\phi}\theta(3\sin\theta-\sin 3\theta)d\theta\ ,\qquad \text{by using }\ \sin^3\theta=\frac{3\sin\theta-\sin 3\theta}{4}$$

$$\text{here,}:\phi=\sin^{-1}\left(\frac{1}{\sqrt{12}}\right)=\cos^{-1}\left(\frac{\sqrt{11}}{\sqrt{12}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{11}}\right)$$

$$\Rightarrow I=8\left[\theta\left(\frac{\cos 3\theta}{3}-3\cos\theta\right)\right]_0^{\phi}-8\int_0^{\phi}\left(\frac{\cos 3\theta}{3}-3\cos\theta\right)d\theta$$

$$=8\phi\left(\frac{\cos 3\phi}{3}-3\cos\phi\right)-8\left(\frac{\sin 3\phi}{9}-3\sin\phi\right)=$$

$$\text{Now use the identity }\ \cos 3\phi=4\cos^3\phi-3\cos\phi\ \text{ and } \sin 3\phi=3\sin\phi-4\sin^3\phi$$

$$\Rightarrow I=\frac{32}{9}(6\sin\phi+\sin^3\phi)-\frac{32\phi}{3}(3\cos\phi-\cos^3\phi)$$

$$=\frac{32}{9}\left(\frac{6}{\sqrt{12}}+\frac{1}{12\sqrt{12}}\right)-\frac{32\phi}{3}\left(\frac{3\sqrt{11}}{\sqrt{12}}-\frac{11\sqrt{11}}{12\sqrt{12}}\right)=\frac{32*73}{9*12\sqrt{12}}-\phi\frac{32*25\sqrt{11}}{3*12\sqrt{12}}$$

$$\Rightarrow I=\frac{4*73}{27\sqrt{3}}-\phi\frac{100\sqrt{11}}{9\sqrt{3}}\quad\Rightarrow\quad I=\frac{4}{27\sqrt{3}}(73-75\phi\sqrt{11})=$$

$$=\frac{4}{27\sqrt{3}}\left(73-75\sqrt{11}\tan^{-1}\left(\frac{1}{\sqrt{11}}\right)\right)$$