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Prove that:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)\Gamma(2n)\Gamma\left(-n+\frac{1}{2}\right)\cos(n\pi)}{8^n\Gamma(n)}=\frac{\sqrt{\pi}}{6}\log_e\left(\frac{3e^{\gamma}}{2}\right)$$

$\gamma \approx 0.577$ *is Euler – Mascheroni's constant*

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$$\text{Let } I=\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)\Gamma(2n)\Gamma\left(-n+\frac{1}{2}\right)\cos(n\pi)}{8^n\Gamma(n)}$$

$$\text{We know: } \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin(n\pi)}\Rightarrow\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(-n+\frac{1}{2}\right)$$

$$=\frac{\pi}{\cos(n\pi)}\ (\text{Euler's reflection formula})\Rightarrow\Gamma\left(-n+\frac{1}{2}\right)\cos(n\pi)=\frac{\pi}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow I=\pi\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)}{8^n}\left(\frac{\Gamma(2n)}{\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)}\right)$$

$$\text{Now, we know }\prod_{k=1}^{m-1}\Gamma\left(x+\frac{k}{m}\right)=\frac{\left(\sqrt{2\pi}\right)^{m-1}}{\sqrt{m}}\frac{1}{m^{mx-1}}\frac{\Gamma(mx)}{\Gamma(x)}\quad\text{for } m\geq 2$$

$$\underbrace{\Rightarrow}_{m=2}\Gamma\left(x+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2x-1}}\frac{\Gamma(2x)}{\Gamma(x)}\Rightarrow\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)\quad(\text{legendre's duplication formula})$$

$$\Rightarrow\frac{\Gamma(2n)}{\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)}=\frac{2^{2n-1}}{\sqrt{\pi}}\Rightarrow I=\pi\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)}{8^n}\left(\frac{2^{2n-1}}{\sqrt{\pi}}\right)=\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)}{2^n}=\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^n\psi(n)$$

Now use, $\psi(n)=-\gamma+H_{n-1}$ (*property of digamma function*)

$$\Rightarrow I=\sqrt{\pi}\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^{n+1}\gamma-\frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}H_{n-1}\left(\frac{-1}{2}\right)^{n-1}$$

$$\text{We know, }-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}\Rightarrow I=-\frac{\ln(1-x)}{1-x}=\sum_{n=1}^{\infty}H_{n-1}x^{n-1}\quad\text{here } H_n=\sum_{k=1}^{n}\frac{1}{k}\text{ and } H_0=0$$

$$\Rightarrow I=\sqrt{\pi}\gamma\frac{\frac{1}{4}}{1+\frac{1}{2}}+\frac{\sqrt{\pi}}{4}\frac{\ln\left(1+\frac{1}{2}\right)}{1+\frac{1}{2}}=\frac{\gamma\sqrt{\pi}}{6}+\frac{\sqrt{\pi}}{6}\ln\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{6}\left(\gamma+\ln\left(\frac{3}{2}\right)\right)\Rightarrow I=\frac{\sqrt{\pi}}{6}\ln\left(\frac{3e^{\gamma}}{2}\right)$$