ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\sum_{n=1}^{\infty}\frac{\sin(n)\sin(n+1)\sin(n-1)}{n}$$

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$$I=\sum_{n=1}^{\infty}\frac{\sin(n)\sin(n+1)\sin(n-1)}{n}=$$

$$=\frac{\cos(2)}{2}\sum_{n=1}^{\infty}\frac{\sin(n)}{n}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(n)\cos(2n)}{n}=(\frac{\cos(2)}{2}+\frac{1}{4})\sum_{n=1}^{\infty}\frac{\sin(n)}{n}-\sum_{n=1}^{\infty}\frac{\sin(3n)}{4n}$$

$$A=\sum_{n=1}^{\infty}\frac{\sin(n)}{n}=Im\left(\sum_{n=1}^{\infty}\frac{e^{in}}{n}\right)=Im(-\ln(1-e^{i}))=Im\left(\ln\left(\frac{1}{1-e^{i}}\right)\right)$$

$$=Im\left(\ln\left(\frac{1}{\cos(1)+1-i\sin(1)}\right)\right)Im\left(\ln\left(\frac{\cos(1)+1+i\sin(1)}{(1+\cos(1))^2+(\sin(1))^2}\right)\right)$$

$$=Im\left(\ln\left(2\sin(\frac{1}{2})\,e^{i\tan^{-1}\left(\frac{\sin(1)}{1-\cos(1)}\right)}\right)\right)=\tan^{-1}\left(\frac{\sin(1)}{1-\cos(1)}\right)$$

$$=\tan^{-1}\left(\cot\left(\frac{1}{2}\right)\right)=\tan^{-1}\left(\tan\left(\frac{\pi-1}{2}\right)\right)=\left(\frac{\pi-1}{2}\right)$$

$$(\frac{\cos(2)}{2}+\frac{1}{4})\sum_{n=1}^{\infty}\frac{\sin(n)}{n}=(\frac{\cos(2)}{2}+\frac{1}{4})\left(\frac{\pi-1}{2}\right)$$

$$\text{note:}\begin{cases}\{\sin(a)\sin(b)=\frac{1}{2}(\cos(a-b)-\cos(a+b))\}\\ \{\sin(a)\cos(b)=\frac{1}{2}(\sin(a-b)+\sin(a+b))\}\end{cases}$$

$$(\textit{according to above solution})\sum_{n=1}^{\infty}\frac{\sin(3n)}{4n}=\frac{\pi-3}{8}$$

$$\mathrm{I}=\left(\frac{\cos(2)}{2}+\frac{1}{4}\right)\left(\frac{\pi-1}{2}\right)+\frac{3-\pi}{8}=\frac{\pi}{4}\cos(2)+\frac{1-\cos(2)}{4}$$

$$\text{ANSWER}=\frac{\pi\cos(2)+(\sin(1))^2}{4}$$