ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{n\epsilon N}\frac{n+1}{(-4)^{n+1}}\zeta(n+1) = \frac{G}{2}+\frac{\pi^2}{16}-\frac{\pi}{8}-\frac{3}{4}\ln 2$$

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$$\psi(1+x)-\psi(1)=\sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{x+k}=\sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{k}\left(1+\frac{x}{k}\right)^{-1} \quad here\ \psi(x)=digamma\ function$$

Let $|x|<1$

$$\Rightarrow \psi(1+x)+\gamma = \sum_{k=1}\frac{1}{k}-\frac{1}{k}\left(1-\frac{x}{k}+\frac{x^2}{k^2}-\frac{x^3}{k^3}+\dots\dots\right)=\sum_{k=1}\left(\frac{x}{k^2}-\frac{x^2}{k^3}+\frac{x^3}{k^4}+\dots\dots\right)$$

$$= x\zeta(2)-x^2\zeta(3)+x^3\zeta(4)-\dots\dots$$

$$\Rightarrow \psi(1+x) = -\gamma + x\zeta(2)-x^2\zeta(3)+x^3\zeta(4)-\dots\dots$$

$$\Rightarrow -x\psi(1-x) = x\gamma + x^2\zeta(2)+x^3\zeta(3)+x^4\zeta(4)+\dots\dots$$

$$= x\gamma+\sum_{n=1}^{\infty}x^{n+1}\zeta(n+1)$$

$$Differentiate\ w.r.t\ x$$

$$\Rightarrow -\psi(1-x)+x\psi'(1-x) = \gamma+\sum_{n=1}^{\infty}(n+1)x^n\zeta(n+1)$$

$$put\ x=(-1/4)$$

$$\Rightarrow -\psi\left(\frac{5}{4}\right)-\frac{1}{4}\psi'\left(\frac{5}{4}\right)=\gamma+\sum_{n=1}^{\infty}(n+1)\left(\frac{-1}{4}\right)^n\zeta(n+1)$$

$$\Rightarrow \sum_{n\epsilon N}\frac{n+1}{(-4)^{n+1}}\zeta(n+1) = \frac{1}{4}\psi\left(\frac{5}{4}\right)+\frac{\gamma}{4}+\frac{1}{16}\psi'\left(\frac{5}{4}\right)$$

$$Now,\quad \psi\left(\frac{5}{4}\right)=4+\psi\left(\frac{1}{4}\right)=4-\gamma-\frac{\pi}{2}-3\ln 2 \quad (By\ Gauss's\ Digamma\ Theorem)$$

$$Therefore,\quad \frac{1}{4}\psi(\frac{5}{4})+\frac{\gamma}{4}=1-\frac{\pi}{8}-\frac{3}{4}ln2 \quad \text{----------------}(A)$$

$$We\ know, \psi'(x)=\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\dots\dots\dots\dots$$

$$\psi'(\frac{5}{4})=16\left\{\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}+\dots\dots\dots\dots\dots\dots\right\}$$

$$\Rightarrow \frac{1}{16}\psi'(\frac{5}{4})=\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}+\dots\dots\dots\dots\dots$$

$$Now,$$

$$\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\dots\dots\dots=G$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\dots\dots\dots=\left\{1-\frac{1}{2^2}\right\}\zeta(2)=\frac{3}{4}\times\frac{\pi^2}{6}=\frac{\pi^2}{8}$$

$$By\ adding\ both\ series,$$

$$\Rightarrow 2\left\{\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\dots\dots\dots\dots\dots\dots\right\}=G+\frac{\pi^2}{8}$$

$$\Rightarrow \frac{1}{16}\psi'(\frac{5}{4})=\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}+\dots\dots\dots\dots=\frac{G}{2}+\frac{\pi^2}{16}-1 \quad \text{----------------}(B)$$

$$Add\ both\ (A)\ and\ (B)$$

$$\frac{1}{4}\psi\left(\frac{5}{4}\right)+\frac{\gamma}{4}+\frac{1}{16}\psi'\left(\frac{5}{4}\right)=\frac{1}{4}\left(4-\gamma-\frac{\pi}{2}-3ln2\right)+\frac{\gamma}{4}+\frac{G}{2}+\frac{\pi^2}{16}-1=1-\frac{\pi}{8}-\frac{3}{4}ln2+\frac{G}{2}+\frac{\pi^2}{16}-1$$

$$=\frac{G}{2}+\frac{\pi^2}{16}-\frac{\pi}{8}-\frac{3}{4}ln2$$

$$\Rightarrow \sum_{n\epsilon N}\frac{n+1}{(-4)^{n+1}}\zeta(n+1)=\frac{G}{2}+\frac{\pi^2}{16}-\frac{\pi}{8}-\frac{3}{4}ln2$$

$here\ \ G=Catalan's\ constant\ \ and\ \ \gamma=Euler-Mascheroni's\ \ constant$