ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\sum_{n\in N} \frac{\sin(n-1)\sin(n)\sin(n+1)}{n}$$

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$$S = \sum_{n\in N} \frac{\sin(n-1)\sin(n)\sin(n+1)}{n}$$

$$\textit{This summ}: S(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}, \quad \textit{where} \ \in (0, 2\pi) \textit{ is well} - \textit{known}$$

$$S(1) = \frac{\pi - 1}{2}, S(3) = \frac{\pi - 3}{2}$$

$$\textit{Since}: \sin(n-1)\sin(n+1) = \frac{1}{2}(\cos(2) - 1 + 1 - \cos(2n))$$

$$S = \frac{1}{2}(\cos(2) - 1)\sum_{n=1}^{\infty} \frac{\sin(n)}{n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin(n)(1 - \cos(2n))}{n} = -sin^2(1)\left(\frac{\pi - 1}{2}\right) + \sum_{n=1}^{\infty} \frac{sin^3(n)}{n} =$$

$$= -sin^2(1)\left(\frac{\pi - 1}{2}\right) + \sum_{n=1}^{\infty} \frac{\frac{1}{4}(3\sin(n) - \sin(3n))}{n} = -sin^2(1)\left(\frac{\pi - 1}{2}\right) + \frac{3}{4}S(1) - \frac{1}{4}S(3) =$$

$$= -sin^2(1)\left(\frac{\pi - 1}{2}\right) + \frac{\pi}{4} = \frac{\pi cos(2) + 2sin^2(1)}{4}$$