ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n}{2^{n+k}(n+k)^3}=-\frac{1}{8}Li_3\left(\frac{1}{4}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n}{2^{n+k}(n+k)^3}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\int_0^1 x^{n-1}log^2(x)\underbrace{\sum_{k=1}^{\infty}\frac{x^k}{2^k}}_{GS}dx$$

$$\frac{1}{2}\int_0^1\frac{log^2(x)}{2-x}\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{2^n}dx=-\frac{1}{2}\int_0^1\frac{xlog^2(x)}{(2-x)(2+x)}dx=$$

$$-\frac{1}{2}\int_0^1\frac{xlog^2(x)}{4-x^2}dx=-\frac{1}{8}\int_0^1\frac{xlog^2(x)}{1-(\frac{x}{2})^2}dx=-\frac{1}{8}\sum_{n=0}^{\infty}\frac{1}{4^n}\int_0^1 x^{2n+1}log^2(x)dx=$$

$$-\frac{2}{16.4}\sum_{n=0}^{\infty}\frac{1}{4^n(n+1)^3}=-\frac{1}{32}\sum_{n=1}^{\infty}\frac{4}{4^n n^3}=-\frac{1}{8}Li_3\left(\frac{1}{4}\right)$$