ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k}}{2^{n+k}(n+k)^3}=Li_2\left(-\frac{1}{2}\right)-Li_3\left(-\frac{1}{2}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k}}{2^{n+k}(n+k)^3}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\int_0^1 x^{n+k-1}log^2(x)dx=$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\int_0^1 x^{k-1}log^2(x)\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{2^n}}_{GS}dx$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\int_0^1\left(-\frac{\frac{x}{2}}{1+\frac{x}{2}}\right)x^{k-1}log^2(x)dx=$$

$$-\frac{1}{2}\int_0^1\frac{log^2(x)}{x+2}\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{2^k}dx=\frac{1}{2}\int_0^1\frac{xlog^2(x)}{(x+2)^2}dx;\ \ f(x)=\sum_{n=1}^{\infty}(-\frac{x}{2})^n=-\frac{x}{x+2}$$

$$*\left\{\frac{\partial}{\partial x}f(x)=\sum_{n=1}^{\infty}\frac{n(-1)^n x^{n-1}}{2^n}=-\frac{2}{(x+2)^2}\ \ \sum_{n=1}^{\infty}\frac{n(-1)^n x^n}{2^n}=-\frac{2x}{(x+2)^2}\right\}$$

$$-\frac{1}{4}\sum_{n=1}^{\infty}\frac{n(-1)^n}{2^n}\int_0^1 x^n log^2(x)dx\overset{IBP}{=}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{n(-1)^n}{2^n(n+1)^3}=$$

$$-\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^3}\right)=-\frac{1}{2}\left(-2Li_2\left(-\frac{1}{2}\right)-1+2Li_3\left(-\frac{1}{2}\right)+1\right)=$$

$$=Li_2\left(-\frac{1}{2}\right)-Li_3\left(-\frac{1}{2}\right)$$

Solution 2 by Ankush Kumar Parcha-India

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k}}{2^{n+k}(n+k)^3}=\sum_{m\in N}(-1)^n\frac{(m-1)}{2^m m^3}=$$

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$$\sum_{m\in N}(-\frac{1}{2})^m.\frac{1}{m^2}-\sum_{m\in N}(-\frac{1}{2})^m.\frac{1}{m^3}=Li_2\left(-\frac{1}{2}\right)-Li_3(-\frac{1}{2})$$

$$\textbf{Note}:\begin{cases}\displaystyle\sum_{a,b\in N}f(a+b)=\sum_{n\in N}(n-1)f(n)\\ \textbf{And},\ \displaystyle\sum_{a,b\in N}f(a+b)=\sum_{a\in N}\sum_{b\in N}f(a+b)\end{cases}$$