ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\frac{1}{x^2y^2(y^2+1)(x+1)^2}$$

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$$\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\frac{1}{x^2y^2(y^2+1)(x+1)^2}=\sum_{x=1}^{\infty}\frac{1}{x^2(x+1)^2}\sum_{y=1}^{\infty}\frac{1}{y^2(y^2+1)}=$$

$$\sum_{x=1}^{\infty}\left(\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{2}{x-1}-\frac{2}{x}\right)\sum_{y=1}^{\infty}\left(\frac{1}{y^2}-\frac{1}{y^2+1}\right)=$$

$$\left(\sum_{x=1}^{\infty}\left(\frac{1}{x^2}+\frac{1}{(x+1)^2}\right)+\underbrace{\sum_{x=1}^{\infty}\left(\frac{2}{x+1}-\frac{2}{x}\right)}_{Telescoping\ \ sum}\right)\sum_{y=1}^{\infty}\left(\frac{1}{y^2}-\frac{1}{y^2+1}\right)=$$

$$\left(-1+\frac{\pi^2}{3}-2\right)\left(\frac{\pi^2}{6}-\sum_{y=1}^{\infty}\frac{1}{y^2+1}\right)=\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\left(\frac{1}{2}\sum_{y=-\infty}^{\infty}\frac{1}{y^2+1}-\frac{1}{2}\right)\right)=$$

$$\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\left(\frac{1}{2}\left(-\pi Res\left(\frac{1}{1+z^2}\cot(\pi z),z=\pm i\right)\right)-\frac{1}{2}\right)\right)=$$

$$\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\left(\frac{\pi}{2}\cot(\pi)-\frac{1}{2}\right)\right)=\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\frac{\pi}{2}\cot(\pi)+\frac{1}{2}\right)=$$

$$\frac{1}{18}(\pi^2-9)(\pi^2-3\pi\cot(\pi)+3)$$