ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\Omega_{a>0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{a^{\alpha n+\beta m}(m+1)^k} = -\sum_{n=1}^{\infty}\left(\frac{-1}{a^{\alpha}}\right)^n\sum_{m=1}^{\infty}\frac{1}{(m+1)^k}\left(\frac{1}{a^{\beta}}\right)^{m+1} =$$

$$\left(1-\sum_{n=0}^{\infty}\left(-\frac{1}{a^{\alpha}}\right)^n\right)a^{\beta}\left(-\frac{1}{a^{\beta}}+\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\left(\frac{1}{a^{\beta}}\right)^{m+1}\right)$$

$$=\left(1-\frac{1}{1+\frac{1}{a^{\alpha}}}\right)\left(-1+a^{\beta}Li_k\left(\frac{1}{a^{\beta}}\right)\right)=\frac{1}{a^{\alpha}+1}\left(a^{\beta}Li_k\left(\frac{1}{a^{\beta}}\right)-1\right)$$

$$As: a=2, \quad \alpha=\beta=1, \quad k=3.$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3}=\frac{1}{2+1}\left(2Li_3\left(\frac{1}{2}\right)-1\right)$$

$$=\frac{1}{3}\left\{2\left(\frac{7}{8}\zeta(3)+\frac{ln^3(2)}{6}-\frac{\pi^2\ln(2)}{12}\right)-1\right\}=$$

$$\frac{7}{12}\zeta(3)+\frac{ln^3(2)}{9}-\frac{\pi^2\ln(2)}{18}-\frac{1}{3}$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3}=-\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\sum_{n=1}^{\infty}\frac{1}{2^m(m+1)^3}=\frac{\frac{1}{2}}{1+\frac{1}{2}}\sum_{m=1}^{\infty}\frac{2}{2^{m+1}(m+1)^3}=$$

$$\frac{2}{3}\left(\sum_{m=1}^{\infty}\frac{1}{2^m m^3}-\frac{1}{2}\right)=\frac{2}{3}Li_3\left(\frac{1}{2}\right)-\frac{1}{3}=\frac{7}{12}\zeta(3)+\frac{ln^3(2)}{9}-\frac{\pi^2\ln(2)}{18}-\frac{1}{3}$$

Solution 3 by Pham Duc Nam-Vietnam

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}(2\sum_{m=1}^{\infty}\frac{1}{2^{m+1}(m+1)^3})=\frac{1}{3}\left(2\sum_{m=2}^{\infty}\frac{1}{2^m m^3}\right)=$$

$$\frac{1}{3}\left(2\left(\sum_{m=1}^{\infty}\frac{1}{2^m m^3}-\frac{1}{2}\right)\right)=\frac{2}{3}\sum_{m=1}^{\infty}\frac{1}{2^m m^3}-\frac{1}{3}$$

$$*\sum_{m=1}^{\infty}\frac{1}{2^m m^3}=\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{2^m}\int_0^1 x^{m-1}ln^2(x)dx=\frac{1}{2}\int_0^1 ln^2(x)dx\left(\sum_{m=1}^{\infty}\frac{x^{m-1}}{2^m}\right)$$

$$=\frac{1}{2}\int_0^1\frac{ln^2(x)}{2-x}dx=$$

$$\frac{1}{2}\int_0^1\frac{ln^2(1-x)}{1+x}dx=\frac{1}{2}I$$

$$*I=\int_0^1\frac{ln^2(1-x)}{1+x}dx,\qquad J=\int_0^1\frac{ln^2(1+x)}{1+x}dx=\frac{1}{3}ln^2(2)$$

$$I-J=\int_0^1\frac{ln^2(1-x)-ln^2(1+x)}{1+x}dx=\int_0^1\frac{\ln\left(\frac{1-x}{1+x}\right)\ln(1-x^2)}{1+x}dx\,,x\to\frac{1-x}{1+x}=$$

$$\int_0^1\frac{\ln(x)\ln\left(\frac{4x}{(1+x)^2}\right)}{1+x}dx=\int_0^1\frac{\ln(x)(2\ln(2)+\ln(x)-2\ln(1+x))}{1+x}dx=2\ln(2)\underbrace{\int_0^1\frac{\ln(x)}{1+x}dx}_{-\frac{\pi^2}{12}}+$$

$$\underbrace{\int_0^1\frac{ln^2(x)}{1+x}dx}_{\frac{3}{2}\zeta(3)}-2\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}dx$$

$$=-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)+2\sum_{n=0}^{\infty}(-1)^n H_n\int_0^1 x^n\ln(x)\,dx=$$

$$=-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)-2\sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(n+1)^2}=-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)+2\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_n-\frac{1}{n}\right)}{n^2}$$

$$=$$

$$-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)+2\left(-\frac{5}{8}\zeta(3)+\frac{3}{4}\zeta(3)\right)=-\frac{\pi^2}{6}\ln(2)+\frac{7}{4}\zeta(3)\ \Rightarrow$$

$$I=\frac{7}{4}\zeta(3)-\frac{\pi^2}{6}\ln(2)+\frac{ln^2(2)}{3}\ \Rightarrow$$

$$S=\frac{1}{3}\left(\frac{7}{4}\zeta(3)-\frac{\pi^2}{6}\ln(2)+\frac{ln^2(2)}{3}\right)-\frac{1}{3}=\frac{7}{12}\zeta(3)+\frac{ln^3(2)}{9}-\frac{\pi^2\ln(2)}{18}-\frac{1}{3}$$

Solution 4 by Ankush Kumar Parcha-India

$$\sum_{n\in N}x^n=\frac{x}{1-x}\quad,\qquad |x|<1\ \Rightarrow\ \sum_{n,m\in N}\frac{(-1)^{n+1}}{2^{n+m}(m+1)^3}\underbrace{=}_{m\to m-1}\frac{2}{3}\left(-\frac{1}{2}+\sum_{n,m\in N}\frac{1}{2^m m^3}\right)=$$

$$-\frac{1}{3}+Li_3\left(\frac{1}{2}\right)\quad (1)$$

$$\left(Li_2(1-z)+Li_2\left(\frac{z-1}{z}\right)=-\frac{ln^2(z)}{2},\qquad z\in C\backslash\{0\}\right)$$

Divide both sides by $-z(1-z)$ *and integrate if with respect to* z. *We get.*

$$-\int\frac{Li_2(1-z)}{z(1-z)}dz-\int Li_2\left(\frac{z-1}{z}\right)\frac{dz}{z(1-z)}=\frac{1}{2}\int\frac{ln^2(z)}{z(1-z)}dz \Rightarrow$$

$$\int d\,Li_3\left(\frac{z-1}{z}\right)-\underbrace{\int\frac{Li_2(1-z)}{z}dz}_{IBP}-\int\frac{Li_2(1-z)}{1-z}dz$$

$$=\frac{1}{2}\int\frac{ln^2(z)}{z}dz+\frac{1}{2}\int\frac{ln^2(z)}{1-z}dz\,Li_3\left(\frac{z-1}{z}\right)+\int d\,Li_3(1-z)$$

$$-Li_2(1-z)\ln(z)+\frac{1}{2}\underbrace{\int\frac{ln^2(z)}{1-z}}_{IBP}dz=\int\frac{d\,ln^3(z)}{6}\Rightarrow$$

$$Li_3\left(\frac{z-1}{z}\right)+Li_3(1-z)-Li_2(1-z)\ln(z)-\frac{ln^2(z)\ln(1-z)}{2}+\underbrace{\int\frac{ln(z)\ln(1-z)}{z}dz}_{IBP}$$

$$=\frac{ln^3(z)}{6}$$

$$\Rightarrow Li_3\left(\frac{z-1}{z}\right)+Li_3(1-z)-Li_2(1-z)\ln(z)-Li_2(z)\ln(z)+Li_3(z)\Rightarrow$$

$$\frac{ln^3(z)}{6}+\frac{ln^2(z)\ln(1-z)}{2}+C$$

$$\left(*\,Li_2(z)+Li_2(1-z)=\frac{\pi^2}{6}-\ln(z)\ln(1-z),z\in C\right)\Rightarrow$$

$$Li_3(z)+Li_3(1-z)+Li_3\left(\frac{z-1}{z}\right)=\frac{\pi^2}{6}\ln(2)+\frac{ln^3(z)}{6}-\frac{ln^2(z)\ln(1-z)}{2}+C\overset{set\,z=1}{\Longrightarrow}$$

$$Li_3(1)+0=-\underbrace{\lim_{x\to1}\frac{ln^2(z)\ln(1-z)}{2}}_{=0}+C\Rightarrow C=\zeta(3)$$

$$Li_3(z)+Li_3(1-z)+Li_3\left(1-\frac{1}{z}\right)=\zeta(3)+\frac{\pi^2}{6}\ln(2)+\frac{ln^3(z)}{6}-\frac{ln^2(z)\ln(1-z)}{2}\overset{at\,z=\frac{1}{2}}{\Longrightarrow}$$

$$Li_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln(2)+\frac{ln^3(2)}{6}$$

Put the value of $Li_3\left(\frac{1}{2}\right)$ *in equation* $-$ (1). *We get:*

$$\Rightarrow -\frac{1}{3}+\frac{2}{3}\left(\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln(2)+\frac{ln^3(2)}{6}\right)\Rightarrow\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3}$$

$$=\frac{7}{12}\zeta(3)+\frac{ln^3(2)}{9}-\frac{\pi^2\ln(2)}{18}-\frac{1}{3}$$

Note : $\zeta(3)\to$ *Apery's constant*