ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\Omega=\sum_{k=1}^{\infty}\frac{1+k^2}{2^k k^2\binom{1+k^2}{k^2}}=Li_2\left(\frac{1}{2}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Mohammad Rostami-Afghanistan

$$\Omega=\sum_{k=1}^{\infty}\frac{1+k^2}{2^k k^2\binom{1+k^2}{k^2}}=\sum_{k=1}^{\infty}\frac{1+k^2}{2^k k^2\frac{(1+k^2)!}{k^2!}}=\sum_{k=1}^{\infty}\frac{1+k^2}{2^k k^2(1+k^2)}=\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{k^2}=$$

$$-\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\int_0^1 x^{k-1}\ln(x)\,dx=-\int_0^1\frac{\ln(x)}{x}\sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^k dx=-\int_0^1\frac{\ln(x)}{x}\frac{\frac{x}{2}}{1-\frac{x}{2}}dx=$$

$$-\int_0^1\frac{\ln(x)}{2-x}dx=-\int_0^1\frac{\ln(1-x)}{1+x}dx\ \left\{x=\frac{1-t}{1+t},\right.$$

$$\left.dx=\frac{-1-t-1+t}{(1+t)^2}dt\rightarrow dx=-\frac{2}{(1+t)^2}dt\right\}$$

$$\Omega=-\int_0^1\frac{\ln\left(1-\frac{1-t}{1+t}\right)}{1+\frac{1-t}{1+t}}\frac{-2}{(1+t)^2}dt=\int_0^1\frac{\ln(1+t)-\ln(2)-\ln(t)}{1+t}dt=$$

$$\int_0^1\frac{\ln(1+t)}{1+t}dt-\ln(2)\int_0^1\frac{1}{1+t}dt-\int_0^1\frac{\ln(t)}{1+t}dt,\qquad \ln(1+t)=u$$

$$=\int_0^{\ln(2)}u\,du-\ln(2)\left[\ln(1+t)\right]\Big|_0^1-\int_0^1\sum_{n=0}^{\infty}(-t)^n\frac{\partial}{\partial a}\Big|_{a=0}t^a dt=\left[\frac{u^2}{2}\right]\Bigg|_0^{\ln(2)}-ln^2(2)-$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{\partial}{\partial a}\Big|_{a=0}\int_0^1 t^{n+a}dt=\frac{ln^2(2)}{2}-ln^2(2)-\sum_{n=0}^{\infty}(-1)^n\frac{\partial}{\partial a}\Big|_{a=0}\frac{1}{n+a+1}=$$

$$-\frac{ln^2(2)}{2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=-\frac{ln^2(2)}{2}+\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^2}=-\frac{ln^2(2)}{2}+\eta(2)=$$

$$-\frac{ln^2(2)}{2}+(1-2^{-1})\zeta(2)=-\frac{ln^2(2)}{2}+\frac{1}{2}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{12}-\frac{1}{2}ln^2(2)=Li_2\left(\frac{1}{2}\right)$$

ROMANIAN MATHEMATICAL MAGAZINE

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\sum_{k=1}^{\infty}\frac{1+k^2}{2^k k^2\binom{1+k^2}{k^2}}$$

$$\Omega_{a,m,n}=\sum_{k=1}^{\infty}\frac{b^2+k^n}{a^k k^m\binom{b^2+k^n}{k^n}}=\sum_{k=1}^{\infty}\frac{b^2+k^n}{a^k k^m\frac{(b^2+k^n)!}{(k^n)!\,(b^2)!}}=$$

$$(b^2)!\sum_{k=1}^{\infty}\frac{b^2+k^n}{a^k k^m\frac{(b^2+k^n)(k^n)!}{(k^n)!}}=(b^2)!\sum_{k=1}^{\infty}\frac{1}{a^k k^m}=(b^2)!\sum_{k=1}^{\infty}\frac{\left(\frac{1}{a}\right)^k}{k^m}=(b^2)!\,Li_m\left(\frac{1}{a}\right)$$

Let $a=2\,,b=1\,,n=m=2:$

$$\Omega_{2,2,2}=\sum_{k=1}^{\infty}\frac{1+k^2}{2^k k^2\binom{1+k^2}{k^2}}=(1^2)!\,Li_2\left(\frac{1}{2}\right)=Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}ln^2(2)$$

$$\boxed{\sum_{k=1}^{\infty}\frac{1+k^2}{2^k k^2\binom{1+k^2}{k^2}}=Li_2\left(\frac{1}{2}\right)}$$