ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$S=\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{(2k^2+k-1)(2k-1)}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pratham Prasad-India

$$\textit{By Partial Fractions}:$$

$$S=\frac{5}{18}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}+\frac{1}{6}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k}$$

$$S=\frac{5}{18}\sum_{k=1}^{\infty}(-1)^k\int_0^1 x^{2k-2}dx+\frac{1}{6}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k}$$

$$\textit{changing order of summation and integration},$$

$$S=-\frac{5}{18}\int_0^1\sum_{k=1}^{\infty}(-x^2)^{k-1}\,dx-\frac{1}{6}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k}$$

$$S=-\frac{5}{18}\int_0^1\frac{1}{1+x^2}dx-\frac{1}{6}G+\frac{1}{9}\ln(2)-\frac{1}{9}$$

$$S=-\frac{5\pi}{72}-\frac{1}{6}G+\frac{1}{9}\ln(2)-\frac{1}{9}$$