If $a, b, c > 0, c \leq 3, 2(ab + bc + ca) \geq \sqrt{(9 + 3abc)(\sqrt{a} + \sqrt{b} + \sqrt{c})}$ and $a^3 + b^3 + c^3 = 3(3c^2 + abc - 3ab - 9c + 9)$, then $a, b, c = ?$

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Solution by Soumava Chakraborty-Kolkata-India

$$a^3 + b^3 + c^3 = 3(3c^2 + abc - 3ab - 9c + 9)$$
$$\Rightarrow a^3 + b^3 + (c^3 - 9c^2 + 27c - 27) + 3ab(3 - c) = 0$$
$$\Rightarrow a^3 + b^3 + (c - 3)^3 + 3ab(3 - c) = 0$$
$$\Rightarrow 3ab(c - 3) + (a + b + c - 3)\begin{pmatrix} a^2 + b^2 + (c - 3)^2 - \\ ab - a(c - 3) - b(c - 3) \end{pmatrix} + 3ab(3 - c) = 0$$
$$\Rightarrow \frac{1}{2}(a + b + c - 3)\left((a - b)^2 + \left(a - (c - 3)\right)^2 + \left(b - (c - 3)\right)^2\right) \overset{(*)}{=} 0$$

Now, $(a - b)^2 + \left(a - (c - 3)\right)^2 + \left(b - (c - 3)\right)^2 = 0 \Rightarrow a = b = c - 3$; but $c - 3 \leq 0 \Rightarrow a, b \leq 0 \to$ contradiction

$$\therefore (a - b)^2 + \left(a - (c - 3)\right)^2 + \left(b - (c - 3)\right)^2 \neq 0 \therefore (*) \Rightarrow$$

$$\boxed{a + b + c - 3 \overset{(**)}{=} 0} \Rightarrow 9 = \frac{\left(\sum_{cyc} a\right)^3}{3} \therefore (9 + 3abc)(\sqrt{a} + \sqrt{b} + \sqrt{c}) \overset{G-H}{\geq}$$

$$\left(\frac{\left(\sum_{cyc} a\right)^3}{3} + 3abc\right)\left(\sum_{cyc} \frac{2a}{a + 1}\right) = \left(\frac{\left(\sum_{cyc} a\right)^3 + 9abc}{3}\right)\left(\sum_{cyc} \frac{2a^2}{a^2 + a}\right) \overset{\text{Bergstrom}}{\geq}$$

$$\left(\frac{\left(\sum_{cyc} a\right)^3 + 9abc}{3}\right)\left(\frac{2\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + \sum_{cyc} a}\right) \overset{a+b+c=3}{=} \left(\frac{\left(\sum_{cyc} a\right)^3 + 9abc}{3}\right)\left(\frac{2\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + \frac{\left(\sum_{cyc} a\right)^2}{3}}\right)$$

$$\overset{?}{\geq} 4(ab + bc + ca)^2$$

$$\Leftrightarrow \left(\left(\sum_{cyc} a\right)^3 + 9abc\right)\left(\sum_{cyc} a\right)^2 \overset{?}{\geq} 2\left(\sum_{cyc} ab\right)^2\left(3\sum_{cyc} a^2 + \left(\sum_{cyc} a\right)^2\right) \overset{a+b+c=3}{\Leftrightarrow}$$

$$\left(\frac{\sum_{cyc} a}{3}\right)\left(\left(\sum_{cyc} a\right)^3 + 9abc\right)\left(\sum_{cyc} a\right)^2 \overset{?}{\geq} 2\left(\sum_{cyc} ab\right)^2\left(3\sum_{cyc} a^2 + \left(\sum_{cyc} a\right)^2\right)$$

$$\Leftrightarrow \boxed{\left(\left(\sum_{cyc} a\right)^3 + 9abc\right)\left(\sum_{cyc} a\right)^3 \underset{(\bullet)}{\overset{?}{\geq}} 6\left(\sum_{cyc} ab\right)^2\left(3\sum_{cyc} a^2 + \left(\sum_{cyc} a\right)^2\right)}$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s - x)(s - y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3) \textit{ and}$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4) \therefore \text{via } (1),(2),(3) \textit{ and } (4), (*)$$

$$\Leftrightarrow (s^3 + 9r^2 s)s^3 \ge 6r^2(4R + r)^2(3(s^2 - 8Rr - 2r^2) + s^2)$$

$$\Leftrightarrow \boxed{s^6 + 9r^2 s^4 - r^2 s^2(384R^2 + 192Rr + 24r^2) + 36r^3(4R + r)^3 \overset{(\bullet\bullet)}{\ge} 0} \textit{ and}$$

$$\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (\bullet\bullet), \text{it suffices to prove :}$$

$$\text{LHS of } (\bullet\bullet) \ge (s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (48Rr - 6r^2)s^4 - r^2 s^2(1152R^2 - 288Rr + 99r^2)$$

$$+r^3(6400R^3 - 2112R^2 r + 1632Rr^2 - 89r^3) \overset{(\bullet\bullet\bullet)}{\ge} 0 \textit{ and}$$

$$\because (48Rr - 6r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (\bullet\bullet\bullet),$$

$$\text{it suffices to prove : LHS of } (\bullet\bullet\bullet) \ge (48Rr - 6r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (384R^2 - 384Rr - 39r^2)s^2 \overset{(\bullet\bullet\bullet\bullet)}{\ge} r(5888R^3 - 7104R^2 r + 528Rr^2 - 61r^3)$$

$$\text{Now, } (384R^2 - 384Rr - 39r^2)s^2 \overset{\text{Gerretsen}}{\ge}$$

$$(384R^2 - 384Rr - 39r^2)(16Rr - 5r^2) \overset{?}{\ge} r(5888R^3 - 7104R^2 r + 528Rr^2 - 61r^3)$$

$$\Leftrightarrow 4t^3 - 15t^2 + 12t + 4 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (4t + 1)(t - 2)^2 \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \therefore$$

$$\boxed{\sqrt{(9 + 3abc)(\sqrt{a} + \sqrt{b} + \sqrt{c})} \ge 2(ab + bc + ca) \text{ with } " = " \text{ iff } a = b = c} \to (1)$$

$$\text{But, } 2(ab + bc + ca) \ge \sqrt{(9 + 3abc)(\sqrt{a} + \sqrt{b} + \sqrt{c})}$$

$$\therefore 2(ab + bc + ca) = \sqrt{(9 + 3abc)(\sqrt{a} + \sqrt{b} + \sqrt{c})} \textit{ and}$$

$$(1) \Rightarrow \boxed{a = b = c \to (***)} \therefore (**), (***) \Rightarrow a = b = c = 1 \ (ans)$$