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JP.166. If $a,b\in[0;+\infty)$ and $n\in\mathbb{N}^*\wedge n\geq 2$ then:

$$(ab)^{\frac{n}{2}}\leq\frac{\sum_{k=0}^{n}a^kb^{n-k}}{n+1}\leq\frac{a^n+b^n}{2}$$

Proposed by Nguyen Van Nho – Nghe An – Vietnam

Solution 1 by Tran Hong-Vietnam

$$(ab)^{\frac{n}{2}}\overset{(1)}{\leq}\frac{b^n+ab^{n-1}+a^2b^{n-2}+..+a^{n-1}b+a^n}{n+1}\overset{(2)}{\leq}\frac{a^n+b^n}{2}$$

Using Cauchy's inequality:

$$\frac{b^n+ab^{n-1}+a^2b^{n-2}+\dots+a^{n-1}b+a^n}{n+1}\geq\sqrt[n+1]{(ab)^{\frac{n(n+1)}{2}}}=(ab)^{\frac{n}{2}}\Rightarrow\textbf{(1) is true.}$$

If $a=b$ then (2) true. If $a\neq b$ (suppose $b>a$) we have

$$\textbf{(2)}\Leftrightarrow\frac{b^{n+1}-a^{n+1}}{(n+1)}\leq\frac{a^n+b^n}{2}(b-a)$$

Let $f(x)=x^n(\forall x\in[a,b])$ and $O(0,0),A(a,0),B(b,0),C(a,f(a)),D(b,f(b))$ we have

$$\frac{b^{n+1}-a^{n+1}}{n+1}=\int_a^b x^n\,dx\leq S_{ABCD}=\frac{1}{2}\cdot(OC+OD)(OB-OA)=\frac{(a^n+b^n)(b-a)}{2}$$

Proved.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $f(x)=x^n$ for all $x\geq 0$ and $n\geq 2$, now $f''(x)=n(n-1)x^{n-2}\geq 0$

Hence f is a convex function, by Hermite – Hadamard

$$\left(\frac{a+b}{2}\right)^n\leq\frac{1}{b-a}\int_a^b x^n\,dx\leq\frac{a^n+b^n}{2}\text{ where }a,b\in[0,\infty)\text{ and }b>a$$

$$\Rightarrow(ab)^{\frac{n}{2}}\overset{AM\geq GM}{\leq}\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\leq\frac{a^n+b^n}{2}$$

$$(ab)^{\frac{n}{2}}\leq\frac{\sum_{k=0}^{n}a^kb^{n-k}}{n+1}\leq\frac{a^n+b^n}{2}$$

(Proved)

Solution 3 by Michael Sterghiou-Greece

$$\textbf{1)}\ \sum_{k=0}^{n}a^kb^{n-k}\geq(n+1)\sqrt[n+1]{\prod_{k=0}^{n}(a^kb^{(n-k)})}=(n+1)\cdot\sqrt[n+1]{a^{\sum_0^n k}\cdot b^{\sum_0^n(n-k)}}=$$

$$(n+1)\cdot\left[(ab)^{\frac{n(n+1)}{2}}\right]^{\frac{1}{n+1}}=(n+1)\cdot(ab)^{\frac{n}{2}}\ \textit{(Left)}$$

2) $(a^n+b^n)(n+1)\ge 2\cdot\sum_{k=0}^{n}a^kb^{n-k}\to\sum_{k=0}^{n}\left(a^n+b^n-a^kb^{n-k}-a^{n-k}b^k\right)\ge 0$

$\to\sum_{k=0}^{n}\left(a^k-b^k\right)\left(a^{n-k}-b^{n-k}\right)\ge 0$ ***which is true as the terms of the sum have the same sign. (Right)***

JP.167. Let $OABC$ be a tetrahedron with $\angle AOB=\angle BOC=\angle COA=90^\circ$ and let P be any point inside the triangle ABC. Denote respectively by d_a, d_b, d_c the distances from P to faces $(OBC),(OCA),(OAB)$. Prove that:

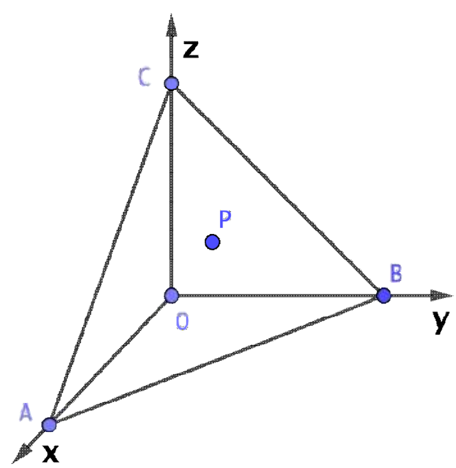
(a) $d_a^2+d_b^2+d_c^2=OP^2$.

(b) $d_ad_bd_c\le\frac{OA\cdot OB\cdot OC}{27}$

(c) $OA\cdot d_a^3+OB\cdot d_b^3+OC\cdot d_c^3\ge OP^4$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Ravi Prakash-New Delhi-India



Let $OA=a\hat{\imath}, OB=b\hat{\jmath}, OC=c\hat{k}, a,b>0$***. Equation of plane*** ABC ***is:*** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Let $P(x,y,z)$ ***be any point in the interior of*** ΔABC***. Then***

(a) $d_a=x, d_b=y, d_c=z$

Now, $d_a^2+d_b^2+b_y^2=x^2+y^2+z^2=OP^2$

$$\text{(b) } \frac{1}{3} = \frac{1}{3}\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) \ge \left(\frac{xyz}{abc}\right)^{\frac{1}{3}}$$

$$\Rightarrow xyz \le 27abc \Rightarrow d_a d_b d_c \le 27(OA)(OB)(OC)$$

$$\text{(c) } ax^3 + by^3 + cz^3 = (ax^3 + by^3 + cz^3)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) =$$

$$= x^4 + y^4 + z^4 + \left(\frac{b}{a}xy^3 + \frac{a}{b}x^3y\right) + \left(\frac{c}{a}xz^3 + \frac{a}{c}xz^3\right) + \left(\frac{b}{c}y^3z + \frac{c}{b}zy^3\right) \ge$$

$$\ge x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 = (x^2 + y^2 + z^2)^2 \Rightarrow$$

$$\Rightarrow (OA)(d_a^3) + (OB)(d_b^3) + (OC)(d_c^3) \ge OP^4$$

JP.168. Let a, b, c be positive real numbers such that:

$$\frac{1}{\sqrt{1 + a^3}} + \frac{1}{\sqrt{1 + b^3}} + \frac{1}{\sqrt{1 + c^3}} \le 1$$

Prove that:

$$a^2 + b^2 + c^2 \ge 12$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{a^3 + 1} = \sqrt{(a + 1)(a^2 - a + 1)}$$

$$\underset{(1)}{\overset{G\le A}{\le}} \frac{(a+1)+(a^2-a+1)}{2} = \frac{a^2+2}{2}, \text{ equality at } a = 2.$$

Similarly, $\sqrt{b^3 + 1} \overset{(2)}{\le} \frac{b^2+2}{2}$, *equality at* $b = 2$ & $\sqrt{c^3 + 1} \overset{(3)}{\le} \frac{c^2+2}{2}$, *equality at* $c = 2$.

$$\text{(1), (2), (3)} \Rightarrow \sum \frac{1}{\sqrt{a^3+1}} \ge 2\sum \frac{1}{a^2+2}$$

$$\overset{Bergstrom}{\ge} \frac{2(1+1+1)^2}{\sum a^2+6} = \frac{18}{\sum a^2+6} \;\&\; \because 1 \ge \sum \frac{1}{\sqrt{a^3+1}}$$

$$\therefore 1 \ge \frac{18}{\sum a^2+6} \Rightarrow \sum a^2 \ge 12, \text{ equality when } a = b = c = 2$$

(proved)

JP.169. Let a, b, c be positive real numbers such that: $a + b + c = 3$.

Prove that:

$$\frac{a^4}{b^4\sqrt{2c(a^3+1)}} + \frac{b^4}{c^4\sqrt{2a(b^3+1)}} + \frac{c^4}{a^4\sqrt{2b(c^3+1)}} \ge \frac{a^2+b^2+c^2}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a^4}{b^4\sqrt{2c(a^3+1)}} + \frac{b^4}{c^4\sqrt{2a(b^3+1)}} + \frac{c^4}{a^4\sqrt{2b(c^3+1)}} \overset{(1)}{\ge} \frac{\sum a^2}{2}$$

$$\textit{(1)} \Leftrightarrow \frac{\left(\frac{a^2}{b^2}\right)^2}{\sqrt{2c(a^3+1)}} + \frac{\left(\frac{b^2}{c^2}\right)^2}{\sqrt{2a(b^3+1)}} + \frac{\left(\frac{c^2}{a^2}\right)^2}{\sqrt{2b(c^3+1)}} \overset{(2)}{\ge} \frac{\sum a^2}{2}$$

$$\textit{Now, } \sqrt{2c(a^3+1)} = \sqrt{\big(c(a+1)\big)(2a^2-2a+2)}$$

$$\underset{(a)}{\overset{G\le A}{\le}} \frac{ca+c+2a^2-2a+2}{2}$$

$$\textit{Similarly, } \sqrt{2a(b^3+1)} \overset{(b)}{\le} \frac{ab+a+2b^2-2b+2}{2} \ \& \ \sqrt{2b(c^3+1)} \overset{(c)}{\le} \frac{bc+b+2c^2-2c+2}{2}$$

(a), (b), (c) ⇒ LHS of (2) ≥

$$2\left[\frac{\left(\frac{a^2}{b^2}\right)^2}{ca+c+2a^2-2a+2} + \frac{\left(\frac{b^2}{c^2}\right)^2}{ab+a+2b^2-2b+2} + \frac{\left(\frac{c^2}{a^2}\right)^2}{bc+b+2c^2-2c+2}\right]$$

$$\overset{Bergstrom}{\ge} \frac{2\left(\sum\frac{a^2}{b^2}\right)^2}{\sum ab + \sum a + 2\sum a^2 - 6 + 6} \quad \left(\because 2\sum a = 6\right)$$

$$= \frac{2\left(\sum\frac{a^2}{b^2}\right)^2}{\sum ab + \frac{1}{3}\left(\sum a\right)^2 + 2\sum a^2} \quad \left(\because \sum a = 3\right)$$

$$= \frac{6\left(\sum\frac{a^2}{b^2}\right)^2}{3\sum ab + \sum a^2 + 2\sum ab + 6\sum a^2} = \frac{6\left(\sum\frac{a^2}{b^2}\right)^2}{7\sum a^2 + 5\sum ab}$$

$$\geq \frac{6\left(\sum \frac{a^2}{b^2}\right)^2}{12\sum a^2}\ \left(\because 5\sum ab \leq 5\sum a^2\right)$$

$$= \frac{\left(\sum \frac{a^2}{b^2}\right)^2}{2\sum a^2} \overset{?}{\geq} \frac{\sum a^2}{2} \Leftrightarrow \sum \frac{a^2}{b^2} \overset{?}{\geq} \sum a^2$$

$$\Leftrightarrow \left(\sum a\right)^2 \left(\sum \frac{a^2}{b^2}\right) \overset{?}{\geq} 9\sum a^2\ \ \left(\because \left(\sum a\right)^2 = 9\right)$$

$$\Leftrightarrow \left(\sum a\right)^2 \left(\frac{\sum a^2b^4}{a^2b^2c^2}\right) \overset{?}{\geq} 9\sum a^2$$

$$\Leftrightarrow \sum a^2b^6 + 2abc\left(\sum ab^4\right) + 2\sum a^7b^5 + 2abc\left(\sum a^2b^3\right) +$$

$$+\sum a^4b^4 \underset{(3)}{\overset{?}{\geq}} 8a^2b^2c^2\left(\sum a^2\right)$$

Now, $\sum a^2b^6 = a^2b^2c^2\left(\frac{b^4}{c^2} + \frac{c^4}{a^2} + \frac{a^4}{b^2}\right)$

$$\underset{(i)}{\overset{Bergstrom}{\geq}} a^2b^2c^2\frac{(\sum a^2)^2}{\sum a^2} = a^2b^2c^2\left(\sum a^2\right)$$

Also, $2abc(\sum ab^4) = 2a^2b^2c^2\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right) = 2a^2b^2c^2\left(\frac{a^4}{ab} + \frac{b^4}{bc} + \frac{c^4}{ac}\right)$

$$\underset{(ii)}{\overset{Bergstrom}{\geq}} 2a^2b^2c^2\frac{(\sum a^2)^2}{\sum ab} \overset{\sum a^2 \geq \sum ab}{\geq} 2a^2b^2c^2\left(\sum a^2\right)$$

Again, $2\sum a^3b^5 + 2abc(\sum a^2b^3) = 2a^2b^2c^2\left(\frac{ab^3}{c^2} + \frac{bc^3}{a^2} + \frac{ca^3}{b^2} + \frac{ab^2}{c} + \frac{bc^2}{a} + \frac{ca^2}{b}\right)$

$$= 2a^2b^2c^2\left[\left(\frac{ab^3}{c^2} + \frac{bc^2}{a}\right) + \left(\frac{bc^3}{a^2} + \frac{ca^2}{b}\right) + \left(\frac{ca^3}{b^2} + \frac{ab^2}{c}\right)\right]$$

$$\underset{(iii)}{\overset{A-G}{\geq}} 2a^2b^2c^2(2b^2 + 2c^2 + 2a^2) = 4a^2b^2c^2\left(\sum a^2\right)$$

Lastly, $\sum a^4b^4 \underset{(iv)}{\overset{\sum x^2 \geq \sum xy}{\geq}} a^2b^2 \cdot b^2c^2 + b^2c^2 \cdot c^2a^2 + c^2a^2 \cdot a^2b^2 =$

$$= a^2b^2c^2\left(\sum a^2\right)$$

(i)+(ii)+(iii) +(iv)⇒(3) is true (proved)

Solution 2 by Michael Sterghiou-Greece

$\sum_{cyc}\frac{a^4}{b^4\sqrt{2c(a^3+1)}} \geq \frac{1}{2}\sum_{cyc} a^2$ ***(1) Let*** $(p,q,r,m) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\sum_{cyc} a^2\right)$

$p = 3, q \leq 3, r \leq 1, m = p^2 - 2q = 9 - 2q$. ***By AM-GM*** $\sqrt{2c(a^3+1)} \leq \frac{1}{2}(2c + a^3 + 1)$

with equality when $c = a = 1$***; same in a cyclic manner. By this and BCS we get***

$\frac{\left(\sum_{cyc}\frac{a}{b^2}\right)^2}{\sum_{cyc}\frac{a^3+2c+1}{2}} \overset{?}{\geq} \frac{m}{2}$ ***(2). For*** $x, y, z > 0$ ***we know that (AM-GM)***

$\sum_{cyc}\frac{x}{y} \geq \left(\sum_{cyc} x\right) \cdot (xyz)^{\frac{1}{3}}$ ***with*** $x = a^2, y = b^2, z = c^2$ ***this*** $\rightarrow \sum_{cyc}\frac{a^2}{b^2} \geq m \cdot r^{-\frac{2}{3}}$

Now, (2) $\rightarrow \frac{2\cdot m^2 r^{-\frac{4}{3}}}{\sum_{cyc} a^3+6+3} \geq \frac{m}{2}$ ***or*** $36r^{-\frac{4}{3}} + 9q - 8qr^{-\frac{4}{3}} - 3r - 36 \geq 0$ ***(3)***

where $\sum_{cyc} a^3 = p^3 - 3pq + 3r = 27 - 9q + 3r$. ***From (3) using the facts***

$q^2 \geq 3pr = 9r$ ***and*** $q \leq \frac{p^3+9r}{4p} = \frac{27+9r}{12}$ ***(Schur) we get the stronger inequality***

$36r^{-\frac{4}{3}} + 27r^{\frac{1}{2}} - 8\cdot\frac{27+9r}{12}\cdot r^{-\frac{4}{3}} - 3r \geq 0$ ***(4). This using the transformation***

$t = r^{\frac{1}{6}}$ ***reduces to:*** $-3t^{14} - 36r^8 + 27t^{11} - 6t^6 + 18 \geq 0$

[$(4) \times r^{\frac{4}{3}}$ ***and*** $r^{\frac{1}{6}} \rightarrow t$***] or*** $3(1-t)\cdot(t^{13} + t^{12} + t^{11} - 8t^{10} - 8t^9 - 8t^8 + 4t^7 + 4t^6 +$

$+6t^5 + 6t^4 + 6t^3 + 6t^2 + 6t + 6 \geq 0$ ***(5). We can observe that*** $6t^3 - 6t^{10} \geq 0$

$2t^7 - 2t^{10} \geq 0$ $(t \leq 1)$ ***and similarly, we work with*** $-8t^9$ ***and*** $-8t^8$***. As the rest in the term*** $t^{13} + t^{12} + \cdots + 6$ ***are all positive we see that (5) holds. Done.***

JP.170. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum value of:

$$P = \frac{x^4}{y^4\sqrt[3]{4z(x^5+1)}} + \frac{y^4}{z^4\sqrt[3]{4x(y^5+1)}} + \frac{z^4}{x^4\sqrt[3]{4y(z^5+1)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Tran Hong-Vietnam

$$\sqrt[3]{4z(x^5+1)}=\sqrt[3]{2z(x+1)\{2x^4-2x^3+2x^2-2x+2\}}$$
$$\overset{(Cauchy)}{\le}\frac{2z+x+1+2x^4-2x^3+2x^2-2x+2}{3}$$
$$=\frac{2z+2x^4-2x^3+2x^2-x+3}{3};$$

Similarly, $\sqrt[3]{4x(y^5+1)}\le\frac{2x+2y^4-2y^3+2y^2-y+3}{3}$;

$$\sqrt[3]{4y(z^5+1)}\le\frac{2y+2z^4-2z^3+2z^2-z+3}{3};$$

$$\therefore\sum\frac{\left(\frac{x^2}{y^2}\right)^2}{\sqrt[3]{4z(x^5+1)}}\overset{(Schwarz)}{\ge}\frac{3\left(\sum\frac{a^2}{b^2}\right)^2}{2\sum a^4-2\sum a^3+2\sum a^2+12}$$

$(\because a=x, b=y, c=z)$

$$=\frac{3}{2}\cdot\frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\left(\sum a^4-\sum a^3+\sum a^2+6\right)}\quad \textbf{(1)}$$

Must show that: $\left(\sum\frac{a^2}{b^2}\right)^2\ge\sum a^4-\sum a^3+\sum a^2+6$. ***But*** $\sum\frac{a^2}{b^2}\overset{(2)}{\ge}\sum a^2$

Must show that: $\left(\sum a^2\right)^2\ge\sum a^4-\sum a^3+\sum a^2+6$

$$\Leftrightarrow 2\sum a^2b^2+\sum a^3-\sum a^2-6\ge 0\ \textbf{(*)}$$

Let $p=a+b+c=3; q=ab+bc+ca, r=abc$;

$$\textbf{(*)}\Leftrightarrow 2(q^2-6r)+(27-9q+3r)-(9-2q)-6\ge 0$$
$$\Leftrightarrow 2q^2-7q-9r+12\ge 0$$
$$\Leftrightarrow (q^2-9r)+(q-3)(q-4)\ge 0\ \textbf{(**)}$$

(**) true because: $q\le\frac{p^2}{3}=3, q^2\ge 9r$.

$$\Rightarrow \textbf{(1)}\ge\frac{3}{2}\cdot 1=\frac{3}{2}.$$

$$\Rightarrow P_{\min}=\frac{3}{2}\Leftrightarrow a=b=c=1.$$

Now, we will prove (2) true:

$$\sum\frac{a^2}{b^2}\ge\sum a^2\Leftrightarrow\left(\sum a\right)^2\left(\frac{\sum a^2b^4}{a^2b^2c^2}\right)\ge 9\sum a^2\left(\because\sum a=3\right)$$

$$\Leftrightarrow \sum a^2b^6 + 2abc\sum ab^4 + 2\sum a^3b^5 + 2abc\sum a^2b^3 + \sum a^4b^4 \ge 8(abc)^2\sum a^2 \quad (3);$$

$$\therefore \sum a^2b^6 = (abc)^2\sum \frac{a^4}{b^4} \overset{(Schwarz)}{\ge} (abc)^2\frac{\left(\sum a^2\right)^2}{\sum a^2} = (abc)^2\sum a^2 \quad (4);$$

$$\therefore 2abc\sum ab^4 = 2(abc)^2\sum \frac{a^4}{ab} \overset{(Schwarz)}{\ge}$$

$$2(abc)^2\frac{\left(\sum a^2\right)^2}{\sum ab} \ge 2(abc)^2\frac{\left(\sum a^2\right)^2}{\sum a^2} = 2(abc)^2\sum a^2 \quad (5);$$

$$\therefore 2\sum a^3b^5 + 2abc\sum a^2b^3 = 2(abc)^2\left\{\sum \frac{ab^3}{c^2} + \sum \frac{ab^2}{c}\right\}$$

$$= 2(abc)^2\left\{\left[\frac{ab^3}{c^2} + \frac{bc^2}{a}\right] + \left[\frac{bc^3}{a^2} + \frac{ca^2}{b}\right] + \left[\frac{ca^3}{b^2} + \frac{ab^2}{c}\right]\right\}$$

$$\overset{(Cauchy)}{\ge} 2(abc)^2(2b^2 + 2c^2 + 2a^2) = 4(abc)^2\sum a^2 \quad (6).$$

$$\therefore \sum a^4b^4 = \sum\{(ab)^2\}^2 \ge (abc)^2\sum a^2 \quad (7);$$

From (4)+(5) +(6) +(7)⇒ (3) true ⇒ (2) true.

JP.171. Let ABC be an acute triangle with perimeter 3. Prove that:

$$\frac{1}{m_a^a} + \frac{1}{m_b^b} + \frac{1}{m_c^c} \ge \frac{3}{R+r}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$LHS \underset{(1)}{\overset{A-G}{\ge}} 3\sqrt[3]{\frac{1}{m_a^a m_b^b m_c^c}}$$

Now, $\sqrt[3]{m_a^a m_b^b m_c^c} = \sqrt[a+b+c]{m_a^a m_b^b m_c^c} \overset{weight\ GM\le AM}{\le} \frac{\sum am_a}{2s} \Rightarrow 3\sqrt[3]{\frac{1}{m_a^a m_b^b m_c^c}} \overset{(2)}{\ge} \frac{3(2s)}{\sum am_a}$

$$\text{(1), (2)} \Rightarrow LHS \ge \frac{3(2s)}{\sum am_a} \overset{?}{\ge} \frac{3}{R+r} \Leftrightarrow \sum am_a \underset{(3)}{\overset{?}{\ge}} 2s(R+r)$$

$$\because m_a \le R(1+\cos A) \text{ etc}, \sum am_a \le \sum a^2 R\cdot 2\frac{s(s-a)}{abc}$$

$$= \frac{2Rs}{4Rrs}\sum a^2(s-a) = \frac{1}{2r}\left(s\sum a^2 - \sum a^3\right)$$

$$=\frac{1}{2r}\{s\cdot 2(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\}$$

$$=\frac{s}{r}(2Rr+2r^2)=2s(R+r)\Rightarrow$$ ***(3) is true (Proved)***

JP.172. Let a, b, c be positive real numbers such that: $abc=1$.

Prove the inequality:

$$\frac{a^4}{b^4\sqrt{a^4+4}}+\frac{b^4}{c^4\sqrt{b^4+4}}+\frac{c^4}{a^4\sqrt{c^4+4}}\geq\sqrt{\frac{3(a+b+c)}{5}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $abc=1$, *give* $a=\frac{x}{y}; b=\frac{y}{z}, c=\frac{z}{x}$

Hence $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c$. *If* $\frac{xy}{z^2}+\frac{yz}{x^2}+\frac{zx}{y^2}\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$

If $(xy)^3+(yz)^3+(zx)^3\geq x^3yz^2+y^3zx^2+z^3xy^2$ *and it's true because*

$$(xy)^3+(xy)^3+(yz)^3\geq 3(x^2y^3z)$$

$$(yz)^3+(yz)^3+(zx)^3\geq 3(y^2z^3x)$$

$$(zx)^3+(zx)^3+(xy)^3\geq 3(z^2x^3y)$$

Hence, similarly $\frac{a^4}{c^4}+\frac{c^4}{b^4}+\frac{b^4}{a^4}\geq\frac{a^3}{c}+\frac{c^3}{b}+\frac{b^3}{a}$

Because $x^{12}+y^{12}+z^{12}\geq x^8yz^3+y^8zx^3+z^8xy^3$ *and* $\frac{a^4}{c^4}+\frac{c^4}{b^4}+\frac{b^4}{a^4}\geq\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}$

Because $x^{12}+y^{12}+z^{12}\geq x^7z^5+y^7x^5+z^7y^5$ *and* $a^5+b^5+c^5\geq\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}$,

$\frac{a^3}{c}+\frac{c^3}{b}+\frac{b^3}{a}$ *consider* $\frac{a^4}{b^4\sqrt{a^4+4}}+\frac{b^4}{c^4\sqrt{b^4+4}}+\frac{c^4}{a^4\sqrt{c^4+4}}\geq\sqrt{\frac{3(a+b+c)}{5}}$

If $\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2\geq\sqrt{\frac{3(a+b+c)}{5}}\left(\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}\right)$

If $\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2\geq\sqrt{\frac{9}{5}(a+b+c)(a^4+b^4+c^4+12)}$

If $\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^4\geq\frac{9}{5}(a+b+c)(a^4+b^4+c^4+12)$

$$\textit{If } \frac{1}{3}\left(\frac{a^8}{b^8}+\frac{b^8}{c^8}+\frac{c^8}{a^8}\right)+30\left(\frac{a^4}{c^4}+\frac{c^4}{b^4}+\frac{b^4}{a^4}\right)+60\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+$$

$$+20\left(\frac{a^6}{b^4c^2}+\frac{b^6}{c^4a^2}+\frac{c^6}{a^4b^2}+\frac{a^4c^2}{b^6}+\frac{a^4b^2}{a^6}+\frac{b^4a^2}{c^6}\right)\geq 9(a^5+b^5+c^5)+$$

$$+9\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^3}{c}+\frac{c^3}{b}+\frac{b^3}{a}\right)+108(a+b+c)$$

$$\textit{If } 5(a^8+b^8+c^8)+40(a^5+b^5+c^5)+15\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right)+15\left(\frac{a^3}{c}+\frac{c^3}{b}+\frac{b^3}{a}\right)+$$

$$+60(a^2+b^2+c^2)\geq 9(a^5+b^5+c^5)+9\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^2}{a}+\frac{a^3}{c}+\frac{c^3}{b}+\frac{b^3}{a}\right)+108(a+b+c)$$

Therefore, it's true.

Solution 2 by Tran Hong-Vietnam

$$LHS=\frac{\left(\frac{a^2}{b^2}\right)^2}{\sqrt{a^4+4}}+\frac{\left(\frac{b^2}{c^2}\right)^2}{\sqrt{b^4+4}}+\frac{\left(\frac{c^2}{a^2}\right)^2}{\sqrt{c^4+4}}\overset{(Schwarz)}{\geq}\frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\sum\sqrt{a^4+4}}\geq\frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\sqrt{3(a^4+b^4+c^4+12)}}$$

Must show that: $\left(\sum\frac{a^2}{b^2}\right)^2\geq\sqrt{\frac{3(a+b+c)}{5}}\sqrt{3(a^4+b^4+c^4+12)}$;

$$\Leftrightarrow 5\left(\sum\frac{a^2}{b^2}\right)^4\geq 9(a+b+c)(a^4+b^4+c^4+12)$$

$$5\{a^{16}c^8+c^{16}b^8+b^{16}a^8\}+20\{c^2a^{10}+a^2b^{10}+b^2c^{10}+c^8a^{10}+a^8b^{10}+b^8c^{10}\}+$$

$$+30\{b^4a^8+a^4c^8+c^4b^8\}+60\{c^2a^4+a^2b^4+b^2c^4\}\geq 9\{a^5+b^5+c^5\}+$$

$$+9\{ab^4+ac^4+ba^4+bc^4+ca^4+cb^4\}+108\{a+b+c\}\ (*)$$

$$\sum a^2\geq\frac{(a+b+c)^2}{3}\geq(a+b+c)\ (\because a+b+c\geq 3\sqrt[3]{abc}=3)$$

$$5\sum a^8\overset{(Chebyshev+Cauchy)}{\geq}5\sum a^5\quad (1)$$

$$\Rightarrow 60\sum a^2b^4\overset{(Chebyshev+Cauchy)}{\geq}60\sum a^2\geq 60\sum a\quad (2)$$

$$36\sum a^5\overset{(Chebyshev)}{\geq}36\cdot\frac{1}{3}\sum a^2\sum a^3\overset{(Cauchy)}{\geq}36\sum a^2\geq 36\sum a\quad (3)$$

$$30\sum a^8b^4\geq 15\sum a^4c+15\sum a^4b=\{9\sum a^4c+9\sum a^4b\}+6\{\sum a^4c+\sum a^4b\}\quad (4)$$

$$6\sum a^4c\overset{(Chebyshev)}{\geq}6\frac{1}{3}\sum a\sum a^4\overset{(Cauchy)}{\geq}6\sum a\quad (5)$$

Same $6\sum a^4b \geq 6\sum a$ **(6)**

$$20\left(\sum a^2b^{10} + \sum a^8b^{10}\right) \overset{(Chebyshev+Cauchy)}{\geq} 40\sum a^5 \quad (7)$$

From (1)+(2) +(3) +(4) +(5) +(6) +(7)⇒ (*) true.

JP.173. Prove that in any triangle ABC,

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \geq \sqrt{\frac{6R}{r}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Lahiru Samarakoon-Sri Lanka

We have to prove, $\left(\because \sum\frac{1}{\sin A} = \frac{s^2R+4Rr}{2sr}\right)$

$$\frac{(s^2 + r^2 + 4Rr)^2}{4s^2r^2} \geq \frac{6R}{r}$$

$$s^4 + r^4 + 16R^2r^2 + 2s^2r^2 + 8r^3R + 8Rrs^2 \geq 24Rrs^2$$

$$s^2(s^2 + 2r^2 - 16Rr) + r^4 + 16R^2r^2 + 8r^3R \geq 0$$

Since, $s^2 \geq 16Rr - 5r^2$***, then we have to prove,***

$$(16Rr - 5r^2)(s^2 + 2r^2 - 16Rr) + r^4 + 16R^2r^2 + 8r^3R \geq 0$$

again, we have to prove,

$$(16Rr - sr^2)^2 + 32Rr^3 - 10r^4 - 256R^2r^2 + 80Rr^3 + r^4 + 16R^2r^2 + 8r^3R \geq 0$$

$$16r^4 + 16R^2r^2 - 40Rr^3 \geq 0$$

$$8r^2(2R^2 - sRr + 2r^2) \geq 0$$

$$8r^2(2R - r)\underbrace{(R - 2r)}_{(+)\,euler}$$

(proved)

Solution 2 by Ruanghaw Chaoka-Chiangrai-Thailand

$$\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)^2 \overset{??}{\geq} \frac{6R}{r}$$

Sine' law; $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2\Delta}$

$$\begin{cases} R = \dfrac{abc}{4\Delta} \\ r = \dfrac{2\Delta}{a+b+c} \end{cases} \Rightarrow \frac{6R}{r} = \frac{3abc(a+b+c)}{4\Delta^2}$$

Now inequality becomes $(ab+bc+ca)^2 \overset{??}{\geq} 3abc(a+b+c)$

$$\because (ab+bc+ca)^2 = (ab)^2+(bc)^2+(ca)^2+2abc(a+b+c)$$

$$\geq abc(a+b+c)+2abc(a+b+c)$$

$= 3abc(a+b+c)$ ***holds at*** $a=b=c$

Solution 3 by Marian Ursărescu-Romania

We must show:

$$\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)^2 \geq \frac{6R}{r} \quad \textbf{(1)}$$

But $\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)^2 \geq 3\sum \frac{1}{\sin A \sin B \sin C}$ ***(2)***

But $\sum \frac{1}{\sin A \sin B} = \frac{2R}{r}$ ***(3)***

From (2)+(3)⇒ $\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)^2 \geq \frac{6R}{r} \Rightarrow$ ***(1) it's true.***

JP.174. Prove that in any triangle ABC,

$$\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c} \geq \sqrt{6(1+\cos A\cos B\cos C)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Lahiru Samarakoon-Sri Lanka

For any $\Delta ABC, \sum \frac{h_a}{a} \geq \sqrt{6(1+\cos A\cos B\cos C)}$

$\prod \cos A = \frac{S^2-4R^2-4Rr-R}{4R^2}$ ***so,***

$$\sum \frac{2\Delta}{a^2} \geq \sqrt{6\left(1+\frac{S^2-4R^2-4Rr-r^2}{4R^2}\right)}$$

$$2\Delta \sum \frac{1}{a^2} \geq \sqrt{\frac{6(S^2-4Rr-r^2)}{4R^2}}$$

$\therefore$ *we have to prove* $\underbrace{4\Delta^2R^2}_{a^2b^2c^2} \times 4\left(\sum\frac{1}{a^2}\right)^2 \geq 3 \times \underbrace{2(S^2-r^2-4Rr)}_{(\sum a^2)}$

$\therefore$ *we have to prove,* $a^2b^2c^2\frac{\left(\sum a^2b^2\right)^2}{\left(a^2b^2c^2\right)^2} \geq 3\left(\sum a^2\right)$

$$\left(\sum a^2b^2\right)^2 \geq 3a^2b^2c^2\left(\sum a^2\right)$$

It's true.

$$\because 3a^2b^2c^2\left(\sum a^2\right) = 3[(a^2b^2)(a^2c^2)+(b^2a^2)(b^2c^2)+(b^2c^2)(a^2c^2)]$$

$\leq 3\frac{\left(a^2b^2+b^2c^2+a^2c^2\right)^2}{3}$. *So, proved*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{h_a}{a} \overset{(1)}{\geq} \sqrt{6(1+\cos A\cos B\cos C)}$$

$$(1) \Leftrightarrow \sum\frac{b^2c^2}{8R^2rs} \geq \sqrt{6\left(1+\frac{s^2-4R^2-4Rr-r^2}{4R^2}\right)}$$

$$\Leftrightarrow \frac{\left(\sum a^2b^2\right)^2}{64R^4r^2S^2} \geq \frac{3\sum a^2}{4R^2} \Leftrightarrow \left(\sum a^2b^2\right)^2 \overset{(2)}{\geq} 3(abc)^2\sum a^2$$

$$(\because 4Rs = abc)$$

Let $s-a=x, s-b=y, s-c=z \therefore a=y+z, b=z+x, c=x+y$

Using the above substitution, (2) becomes (upon simplification):

$$\sum x^8+4\sum x^7y+4\sum xy^7+4\sum x^6y^2+4\sum x^2y^6+12xyz\left(\sum x^5\right)+$$

$$+6xyz\left(\sum x^4y+\sum xy^4\right) \overset{(3)}{\geq} 2\sum x^5y^3+2\sum x^3y^5+5\sum x^4y^4+$$

$$+10xyz\left(\sum x^3y^3+\sum x^2y^3\right)+4x^2y^2z^2\left(\sum x^2\right)+8x^2y^2z^2\left(\sum xy\right)$$

We have, $2\sum x^6y^2+2\sum x^2y^6 \overset{Chebysev}{\geq} \frac{2}{2}\sum x^2y^2(x^2+y^2)(x^2+y^2)$

$$\underset{(a)}{\overset{A-G}{\geq}} 2\sum x^2y^2(x^2+y^2)\cdot xy = 2\sum x^5y^3+2\sum x^3y^5$$

Also, $2\sum x^6y^2+2\sum x^2y^6 \underset{(b)}{\overset{A-G}{\geq}} 4\sum x^4y^4$

Also, $\sum x^7y + \sum xy^7 \overset{A-G}{\underset{(c)}{\geq}} 2\sum x^4y^4$

Again, $12xyz\left(\sum x^5\right) = 6xyz\sum\left(x^5 + y^5\right) \overset{Chebyshev}{\geq} \frac{1}{2}6xyz\sum(x^2 + y^2)(x^3 + y^3)$

$$\overset{A-G}{\underset{(d)}{\geq}} 6xyz\sum xy\cdot xy(x+y) = 6xyz\left(\sum x^3y^2 + \sum x^2y^3\right)$$

Again, $4xyz(\sum x^4y + \sum xy^4) = 4xyz\sum xy\,(x^3 + y^3)$

$$\overset{(e)}{\geq} 4xyz\sum x^2y^2(x+y) = 4xyz\left(\sum x^3y^2 + \sum x^2y^3\right)$$

Moreover, $2xyz(\sum x^4y + \sum xy^4) = 2xyz\{\sum z(x^4 + y^4)\}$

$$\overset{Chebyshev}{\geq} \frac{2xyz}{2}\sum z(x^2+y^2)^2 \overset{A-G}{\underset{(f)}{\geq}} xyz\left\{\sum 2xyz(x^2+y^2)\right\} = 4x^2y^2z^2\left(\sum x^2\right)$$

Also, $\sum x^4y^4 \geq x^2y^2\cdot y^2z^2 + y^2z^2\cdot z^2x^2 + x^2y^2\cdot z^2x^2$

$$= x^2y^2z^2\left(\sum x^2\right) \overset{(g)}{\geq} x^2y^2z^2\left(\sum xy\right)$$

Also, $\sum x^8 \geq \sum x^4y^4 \overset{by\ (9)}{\underset{(h)}{\geq}} x^2y^2z^2(\sum xy)$

Lastly, $3(\sum x^7y + \sum xy^7) = 3\{\sum z\,(z^7 + y^7)\} \overset{Chebyshev}{\geq} \frac{3}{2}\sum z(x^3 + y^3)(x^4 + y^4)$

$$\geq \frac{3}{2}\sum zxy(x+y)(x^4+y^4) \overset{A-G}{\geq} 3xyz\sum(x+y)\,x^2y^2$$

$$= 3xyz\sum\{z^3(x^2+y^2)\} \overset{A-G}{\geq} 3xyz\sum(z^3\cdot 2xy)$$

$$= 6x^2y^2z^2\left(\sum x^2\right) \overset{(j)}{\geq} 6x^2y^2z^2\left(\sum xy\right)$$

(a)+(b) +(c) +(d) +(e) +(f) +(g) +(h) +(j)⇒ (3)

It is true (proved)

JP.175. Prove that in any acute triangle ABC,

$$m_ar_a + m_br_b + m_cr_c \leq s^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Bogdan Fustei-Romania

In any acute-angled ΔABC we have the following inequality:

$$m_a \le 2R\cos^2\frac{A}{2} \text{ (and the analogs)}$$

$$r_a = \frac{S}{p-a} \text{ (and the analogs)}$$

$$r_b + r_c = \frac{S}{p-b} + \frac{S}{p-c} = S\left(\frac{1}{p-b} + \frac{1}{p-c}\right) = \frac{S(p-b+p-c)}{(p-b)(p-c)}$$

$$r_b + r_c = \frac{Sa}{(p-b)(p-c)}; S = \sqrt{p(p-a)(p-b)(p-c)}$$

$$r_b + r_c = \frac{a\sqrt{p(p-a)(p-b)(p-c)}}{(p-b)(p-b)} = a\sqrt{\frac{p(p-a)}{(p-b)(p-c)}}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}} \text{ (and the analogs)}$$

$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}} \text{ (and the analogs)}$$

$$\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}} \text{ (and the analogs)}$$

$$\left.\begin{array}{r} a = 2R\sin A \ (and\ the\ analogs) \\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \ (and\ the\ analogs) \end{array}\right\} \Rightarrow a = 4R\sin\frac{A}{2}\cos\frac{A}{2} \text{ (and the analogs)}$$

$$r_b + r = 4R\sin\frac{A}{2}\cos\frac{A}{2}\cdot\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} = 4R\cos^2\frac{A}{2}$$

$$\frac{r_b + r_c}{2} = 2R\cos^2\frac{A}{2} \text{ (and the analogs)}$$

So, we have the following: $m_a \le 2R\cos^2\frac{A}{2}$ ***(and the analogs)***

$$\Leftrightarrow m_a \le \frac{r_b + r_c}{2} \text{ (and the analogs)}$$

$$m_a r_a \le \frac{r_a(r_b + r_c)}{2} \text{ (and the analogs)}$$

But $r_a r_b + r_b r_c + r_a r_c = p^2$***. Summing we have***

$$m_a r_a + m_b r_b + m_c r_c \le \frac{2p^2}{2} = p^2 \text{ for } \Delta ABC \text{ acute – angled.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because \Delta ABC \text{ is acute}, \therefore m_a \le 2R\cos^2\frac{A}{2}, \text{etc.}, \therefore \sum m_a r_a \le \sum 2R\cos^2\frac{A}{2} s\tan\frac{A}{2}$$

$$= 2Rs\sum\cos\frac{A}{2}\sin\frac{A}{2} = Rs\sum\left(\frac{a}{2R}\right) = \frac{s}{2}\left(\sum a\right) = s^2 \quad \text{(proved)}$$

Solution 3 by Marian Ursărescu-Romania

In any acute ΔABC we have: $m_a \le 2R\cos^2\frac{A}{2}$ (1)

and $r_a = \frac{S}{s-a}$ (2). From (1)+(2)$\Rightarrow \sum m_a r_a \le 2RS\sum\frac{\cos\frac{2A}{2}}{s-a} \Rightarrow$

$$\sum m_a r_a \le 2Rrs\sum\frac{\cos^2\frac{A}{2}}{s-a} \quad (3)$$

But in any ΔABC we have: $\sum\frac{\cos^2\frac{A}{2}}{s-a} = \frac{s}{2Rr}$ (4)

From (3)+(4)$\Rightarrow \sum m_a r_a \le 2Rrs\cdot\frac{s}{2Rr} = s^2$

Solution 4 by Marin Chirciu – Romania

We prove the following lemma:

Lemma 1

2) In acute ΔABC:

$$m_a \le 2R\cos^2\frac{A}{2}$$

Mircea Lascu's inequality

Proof

Let M be the middle of BC side and O the circumcenter of ΔABC. In ΔAMO we have

$$AM \le AO + OM \Leftrightarrow m_a \le R + R\cos A = R(1+\cos A) = R\cdot 2\cos^2\frac{A}{2} = 2R\cos^2\frac{A}{2}$$

Equality holds if and only if $b = c$ or if $A = 90°$.

Back to the main problem:

Using Lemma 1 and $r_a = \frac{S}{s-a}$ we obtain:

$$\sum m_a r_a \le \sum 2R\cos^2\frac{A}{2}\cdot\frac{S}{s-a} = 2RS\sum\frac{\cos^2\frac{A}{2}}{s-a} = 2Rrs\sum\frac{\frac{s(s-a)}{bc}}{s-a} =$$

$$= 2Rrs^2\sum\frac{1}{bc} = 2Rrs^2\cdot\frac{a+b+c}{abc} = 2Rrs^2\cdot\frac{2s}{4Rrs} = s^2.$$

Equality holds if and only if the triangle is equilateral.

Remark.

Let's hightlight an inequality having an opposite sense:

3) In ΔABC:

$$m_a r_a + m_b r_b + m_c r_c \geq 27r^2$$

Proposed by Marin Chirciu – Romania

Solution

We prove the following lemma:

Lemma 2.

4) In ΔABC:

$$m_a \geq \frac{b^2 + c^2}{4R}$$

Tereshin's inequality

Proof

We write the power of M ***point (the middle of*** BC ***side) towards the circumcircle of*** ΔABC:

$$MA \cdot MD = MB \cdot MC \Leftrightarrow m_a(AD - m_a) = \frac{a}{2} \cdot \frac{a}{2} \Leftrightarrow m_a \cdot AD = \frac{a^2}{4} + m_a^2 \Leftrightarrow m_a \cdot AD = \frac{b^2 + c^2}{2}$$

As $AD \leq 2R$ ***it follows*** $m_a \geq \frac{b^2+c^2}{4R}$. ***Equality holds if*** $b = c$ ***or if*** $A = 90°$.

Back to the main problem:

Using Lemma 2 and $r_a = \frac{S}{s-a}$ ***we obtain:***

$$\sum m_a r_a \geq \sum \frac{b^2 + c^2}{4R} \cdot \frac{S}{s-a} = \frac{S}{4R} \sum \frac{b^2 + c^2}{s-a} = \frac{S}{4R} \cdot \frac{2[s^2(2R + 3r) - r(4R + r)^2]}{S} = $$

$$= \frac{s^2(2R+3r) - r(4R+r)^2}{2R} \quad (1)$$

Using (1) is sufficient to prove that:

$\frac{s^2(2R+3r)-r(4R+r)^2}{2R} \geq 27r^2 \Leftrightarrow s^2(2R+3r) \geq r(16R^2+62Rr+r^2)$, ***which follows from***

Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$. ***It remains to prove that:***

$$(16Rr-5r^2)(2R+3r) \geq r(16R^2+62Rr+r^2) \Leftrightarrow 2R^2-3Rr-2r^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(2R+r) \geq 0 \text{ obviously form Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

Remark.

We can prove the double inequality:

1) In acute-angled ΔABC:

$$27r^2 \leq m_a r_a + m_b r_b + m_c r_c \leq s^2.$$

Solution

See inequalities 1) and 3).

Equality holds if and only if the triangle is equilateral.

JP.176. If $a, b > 0$, then:

$$(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x} > \frac{4\sqrt{2}ab}{a+b}, \forall x \in \left(0;\frac{\pi}{2}\right)$$

Proposed by Rovsen Pirguliyev – Sumgait – Azebaijan

Solution 1 by Tran Hong-Vietnam

$$\text{Inequality} \Leftrightarrow (a+b)^2\tan x + 2ab\sin x > 4\sqrt{2}abx$$

$$(a+b)^2\tan x + 2ab\left(\sin x - 2\sqrt{2}x\right) > 0 \quad (*)$$

$$\text{Let } f(x) = (a+b)^2\tan x + 2ab\left(\sin x - 2\sqrt{2}x\right), \left(0 < x < \frac{\pi}{2}\right)$$

$$f'(x) = (a+b)^2\frac{1}{\cos^2 x} + 2ab\left(\cos x - 2\sqrt{2}\right)$$

$$f''(x) = 2(a+b)^2\frac{\sin x}{\cos^3 x} - 2ab\sin x$$

$$= 2\sin x\left(\frac{[a+b]^2}{\cos^3 x} - ab\right) = 2\sin x\left(\frac{[a+b]^2 - ab\cos^3 x}{\cos^3 x}\right)$$

$$\geq 2ab\sin x\left(\frac{2-\cos^3 x}{\cos^3 x}\right) > 0, \forall x \in \left(0,\frac{\pi}{2}\right)$$

$$\Rightarrow f'(x) \nearrow \text{ on } \left(0,\frac{\pi}{2}\right) \Rightarrow f'(x) > f'(0) = (a+b)^2 + 2ab\left(1-2\sqrt{2}\right)$$

$$= a^2 + b^2 + 2ab\left(2-2\sqrt{2}\right) \geq 2ab\left(3-2\sqrt{2}\right) > 0$$

$$\Rightarrow f(x) \nearrow \text{ on } \left(0,\frac{\pi}{2}\right) \Rightarrow f(x) > f(0) = 0 \Rightarrow (*) \text{ true.}$$

Solution 2 by Ravi Prakash-New Delhi-India

For $0 \leq x < \frac{\pi}{2}$, *let* $f(x) = (a+b)^2 \sin x + 2ab\tan x - 4\sqrt{2}abx$

$$f'(x) = (a+b)^2\cos x + 2ab\sec^2 x - 4\sqrt{2}ab \geq 4ab\cos x + 2ab\sec^2 x - 4\sqrt{2}ab$$

$$\geq 6ab\left[(\cos x)^{4ab}(\sec^2 x)^{2ab}\right]^{\frac{1}{6}ab} - 4\sqrt{2}ab \geq 6ab - 4\sqrt{2}ab > 0$$

$\Rightarrow f(x)$ *is an increasing function on* $\left[0,\frac{\pi}{2}\right] \Rightarrow f(x) > f(0)$ *for* $0 < x < \frac{\pi}{2}$

$$\Rightarrow (a+b)^2\sin x + 2ab\tan x > 4\sqrt{2}ab\,x \Rightarrow (a+b)\frac{\sin x}{x} + \frac{2ab}{a+b}\cdot\frac{\tan x}{x} > \frac{4\sqrt{2}ab}{a+b}$$

JP.177. If $a,b,c \geq 0$ then:

$$2(a+b+c) + \sum_{cyc}\sqrt{a^2+b^2-ab} \geq 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kelvin Hong-Rawang-Malaysia

We have: $(a+b+c)(b+c+a) \overset{Cauchy-Schwarz\ Inequality}{\geq} \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2$

$\therefore a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}$. *Also, that*

$$\sum_{cyc}\sqrt{a^2+b^2-ab} \overset{AM-GM}{\geq} \sum_{cyc}\sqrt{2ab-ab} = \sum_{cyc}\sqrt{ab}$$

Therefore

$$2(a+b+c) + \sum_{cyc}\sqrt{a^2+b^2-ab} \geq 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) + \sum_{cyc}\sqrt{ab} = 3\sum_{cyc}\sqrt{ab}$$

Solution 2 by Amit Dutta-Jamshedpur-India

$\because$ *We know that:* $(a^2+b^2-ab) = \frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2$

$$\Rightarrow \sqrt{a^2+b^2-ab}=\sqrt{\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2}\geq\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \sqrt{a^2+b^2-ab}\geq\left(\frac{a+b}{2}\right)$$

$$\sum_{cyc}\sqrt{a^2+b^2-ab}\geq\sum_{cyc}\left(\frac{a+b}{2}\right)\geq\sum_{cyc}a$$

$$\Rightarrow \sum_{cyc}(a+b)+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq\sum_{cyc}(a+b)+\sum_{cyc}a$$

$$\geq 3(a+b+c)\geq\frac{3}{2}\left(\sum 2a\right)\geq\frac{3}{2}\{(a+b)+(b+c)+(c+a)\}$$

$$\overset{AM-GM}{\geq}\frac{3}{2}\left(2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ac}\right)$$

$$2(a+b+c)+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\right)$$

(proved)

Solution 3 by Boris Colakovic-Belgrade-Serbia

$$a^2+b^2-ab\geq ab\Leftrightarrow\sqrt{a^2+b^2-ab}\geq\sqrt{ab}\Leftrightarrow\sum_{cyc}\sqrt{a^2+b^2-ab}\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\quad (1)$$

$$2(a+b+c)=(a+b)+(b+c)+(c+a)\overset{AM-GM}{\geq}2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ca}\quad (2)$$

From (1) and (2) $\Rightarrow$ ***LHS*** $\geq 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$

Solution 4 by Michael Sterghiou-Greece

$$2\sum_{cyc}a+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 3\sum_{cyc}\sqrt{ab}\quad (1)$$

LHS (1) $=2\sum_{cyc}a+\sum_{cyc}\sqrt{2ab-ab}=2\sum_{cyc}a+\sum_{cyc}\sqrt{ab}$

It suffices to show that: $\sum_{cyc}a\geq\sum_{cyc}\sqrt{ab}$ ***or*** $\sum_{cyc}\left(\sqrt{a}\right)^2\geq\sum_{cyc}\sqrt{ab}$

which holds (rearrangement inequality).

Solution 5 by Ravi Prakash-New Delhi-India

$$a+b+\sqrt{a^2+b^2-ab}-3\sqrt{ab}\geq a+b+\sqrt{2ab-ab}-3\sqrt{ab}=$$

$$=a+b-2\sqrt{ab}=\left(\sqrt{a}-\sqrt{b}\right)^2\geq 0$$

$$\Rightarrow a+b+\sqrt{a^2+b^2-ab}\geq 3\sqrt{ab}\quad (1)$$

Similarly, $b+c+\sqrt{b^2+c^2-bc}\geq 3\sqrt{bc}$ ***(2)***

and $c+a+\sqrt{c^2+a^2-ca}\geq 3\sqrt{ca}$ ***(3)***

Adding (1), (2), (3), we get:

$$2(a+b+c)+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Solution 6 by Seyran Ibrahimov-Maasilli-Azerbaijan

$$\sum_{cyc}a+b+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 3\sum_{cyc}\sqrt{ab}$$

$$a+b+\sqrt{a^2+b^2-ab}\geq 3\sqrt{ab}\Rightarrow \textit{(1)}$$

$$\Rightarrow \left(\sqrt{a}-\sqrt{b}\right)^2+\sqrt{a^2+b^2-ab}-\sqrt{ab}\geq 0\ \ (\forall a,b\ (a-b)^2\geq 0)$$

$$\overset{a^2+b^2\geq 2ab}{\Rightarrow}\left(\sqrt{a}-\sqrt{b}\right)^2+\sqrt{ab}-\sqrt{ab}=\left(\sqrt{a}-\sqrt{b}\right)^2\geq 0\quad \textit{(*)}$$

$$\overset{(*)}{\Rightarrow} b+c+\sqrt{b^2+c^2-bc}\geq 3\sqrt{bc}\ \ \textit{(2)}$$

$$\wedge\ a+c+\sqrt{a^2+c^2-ac}\geq 3\sqrt{ac}\ \ \textit{(3)}$$

(1)+(2)+(3)⇒

$$2\sum_{cyc}a+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 3\sum_{cyc}\sqrt{ab}$$

(Proved)

Solution 7 by Tran Hong-Vietnam

Using Cauchy's inequality, we have: $a+b\geq 2\sqrt{ab}$; $b+c\geq 2\sqrt{bc}$; $c+a\geq 2\sqrt{ac}$

$$\rightarrow 2(a+b+c)\geq 2\left(\sqrt{ab}+\sqrt{ac}+\sqrt{bc}\right)\ \ \textit{(1)}$$

$$\sqrt{a^2+b^2-ab}\geq\sqrt{2ab-ab}=\sqrt{ab}\ \ \textit{(2)}$$

$$\sqrt{b^2+c^2-bc}\geq\sqrt{2bc-bc}=\sqrt{bc}\ \ \textit{(3)}$$

$$\sqrt{a^2+c^2-ac}\geq\sqrt{2ac-ac}=\sqrt{ac}\ \ \textit{(4)}$$

→ ***(1)+(2)+(3)+(4) we proved. Equality then*** $a=b=c$.

Solution 8 by Soumava Chakraborty-Kolkata-India

$$2\sum a+\sum\sqrt{a^2+b^2-ab}\overset{(1)}{\geq}3\left(\sum\sqrt{ab}\right)$$

$$\because a^2+b^2-ab=\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2\ge\frac{(a+b)^2}{4}$$

$$\therefore \sqrt{a^2+b^2-ab}\ge\frac{a+b}{2}\ (\because a+b\ge 0\ as\ a,b\ge 0)\ \text{etc.}$$

$$\therefore \text{LHS of (1)} \overset{(a)}{\ge} 2\sum a+\frac{1}{2}\sum(a+b)=3\sum a$$

$$\text{Also, RHS of (1)} \overset{CBS}{\le} 3\sqrt{\sum a}\sqrt{\sum a}=3\sum a \overset{by\ (a)}{\le} \text{LHS of (1) (Proved)}$$

Solution 9 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y\ge 0$, we have $x^2-xy+y^2\ge\left(\frac{x+y}{2}\right)^2$

Hence for $a,b,c\ge 0$, we get

$$2(a+b+c)+\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}$$

$$\ge 2(a+b+c)+\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$

$$=(a+b)+(b+c)+(c+a)+\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$

$$\ge 2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ca}+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}=3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Therefore, it is to be true.

JP.178. If $a,b>0$ then:

$$a^3+b^3+\left(\sqrt{a^2+b^2}\right)^3+\frac{4a^2b^2}{a+b+\sqrt{a^2+b^2}}>4ab\sqrt{a^2+b^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Put $a=r\cos\theta,\ b=r\sin\theta,\ 0<\theta<\frac{\pi}{2}$. The inequality

$$a^3+b^3+\left(\sqrt{a^2+b^2}\right)^3+\frac{4a^2b^2}{a+b+\sqrt{a^2+b^2}}>4ab\sqrt{a^2+b^2}$$

becomes $\cos^3\theta+\sin^3\theta+1+\frac{4\cos^2\theta\sin^2\theta}{\cos\theta+\sin\theta+1}>4\cos\theta\sin\theta$

$$\Leftrightarrow \cos^3\theta+\sin^3\theta+1+\frac{2\cos\theta\sin\theta\left[(\sin\theta+\cos\theta)^2-1\right]}{\cos\theta+\sin\theta+1}>4\cos\theta\sin\theta$$

$$\Leftrightarrow \cos^3\theta + \sin^3\theta + 1 + 2\cos\theta\sin\theta(\cos\theta + \sin\theta - 1) - 4\cos\theta\sin\theta > 0$$

$$\Leftrightarrow 1 + \cos^3\theta + \sin^3\theta + 2\cos^2\theta\sin\theta + 2\cos\theta\sin^2\theta - 6\cos\theta\sin\theta > 0$$

$$\Leftrightarrow 1 + \cos\theta(\cos^2\theta + \sin^2\theta) + \sin\theta(\sin^2\theta + \cos^2\theta) +$$

$$+\cos^2\theta\sin\theta + \sin^2\theta\cos\theta - 6\sin\theta\cos\theta > 0$$

$$\Leftrightarrow 1 + \cos\theta + \sin\theta + \cos^2\theta\sin\theta + \sin^2\theta\cos\theta - 6\sin\theta\cos\theta > 0$$

$$\Leftrightarrow (\sin\theta - \cos\theta)^2 + (\cos\theta + \sin^2\theta\cos\theta - 2\sin\theta\cos\theta) +$$

$$+(\sin^2\theta + \cos^2\theta\sin\theta - 2\sin\theta\cos\theta) > 0$$

$$\Leftrightarrow (\sin\theta - \cos\theta)^2 + \cos\theta(1 - \sin\theta)^2 + \sin\theta(1 - \cos\theta)^2 > 0$$

which is true as at least one factor is positive.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b > 0$***, we have*** $2(a^2 + b^2) \geq 4ab \Rightarrow 4ab + 2(a^2 + b^2) \geq 8ab \Rightarrow$

$$\Rightarrow \left(\sqrt{a^2 + b^2}\right)(a + b) + 2(a^2 + b^2) + \frac{4a^2b^2}{(\sqrt{a^2 + b^2})(a + b)} \geq 8ab$$

$$\Rightarrow \left(\sqrt{a^2 + b^2}\right)(a + b) + 2(a^2 + b^2) + \frac{8a^2b^2}{(\sqrt{a^2 + b^2})(a + b) + (a^2 + b^2)} > 8ab$$

$$\Rightarrow (a^2 + b^2)(a + b) + 2(a^2 + b^2)\sqrt{a^2 + b^2} + \frac{8a^2b^2}{(a + b) + \sqrt{a^2 + b^2}} \geq 8ab\sqrt{a^2 + b^2}$$

$$\Rightarrow \frac{(a^2 + b^2)(a + b)}{2} + (a^2 + b^2)\sqrt{a^2 + b^2} + \frac{4a^2b^2}{(a + b) + \sqrt{a^2 + b^2}} \geq 4ab\sqrt{a^2 + b^2}$$

$$\Rightarrow a^3 + b^3 + \left(\sqrt{a^2 + b^2}\right)^3 + \frac{4a^2b^2}{(a + b) + \sqrt{a^2 + b^2}} \geq 4ab\sqrt{a^2 + b^2}$$

Therefore $a^3 + b^3 + \left(\sqrt{a^2 + b^2}\right)^3 + \frac{4a^2b^2}{(a+b)+\sqrt{a^2+b^2}} > 4ab\sqrt{a^2 + b^2}$ ***(true)***

Solution 3 by Serban George Florin-Romania

$$a^3 + b^3 + (a^2 + b^2)\sqrt{a^2 + b^2} + \frac{4a^2b^2\left(a + b - \sqrt{a^2 + b^2}\right)}{(a + b)^2 - (a^2 + b^2)} > 4ab\sqrt{a^2 + b^2}$$

$$a^3 + b^3 + (a^2 + b^2)\sqrt{a^2 + b^2} + \frac{4a^2b^2\left(a + b - \sqrt{a^2 + b^2}\right)}{2ab} > 4ab\sqrt{a^2 + b^2}$$

$$(a + b)(a^2 - ab + b^2) + (a^2 + b^2)\sqrt{a^2 + b^2} + 2ab(a + b) - 2ab\sqrt{a^2 + b^2} > 4ab\sqrt{a^2 + b^2}$$

$$(a+b)(a^2-ab+b^2+2ab)+(a^2+b^2)\sqrt{a^2+b^2}>6ab\sqrt{a^2+b^2}$$

$$(a+b)(a^2+ab+b^2)+(a^2+b^2)\sqrt{a^2+b^2}>6ab\sqrt{a^2+b^2}|:b^3$$

$$\left(\frac{a}{b}+1\right)\left[\left(\frac{a}{b}\right)^2+\frac{a}{b}+1\right]+\left[\left(\frac{a}{b}\right)^2+1\right]\sqrt{\left(\frac{a}{b}\right)^2+1}>6\left(\frac{a}{b}\right)\sqrt{\left(\frac{a}{b}\right)^2+1},\frac{a}{b}=x,x>0$$

$$(x+1)(x^2+x+1)+(x^2+1)\sqrt{x^2+1}>6x\sqrt{x^2+1}|\frac{(\forall)x>0}{:x\sqrt{x^2+1}}$$

$$\frac{(x+1)(x^2+x+1)}{x\sqrt{x^2+1}}+\frac{(x^2+1)\sqrt{x^2+1}}{x\sqrt{x^2+1}}>6$$

$$\frac{(x+1)x}{x\sqrt{x^2+1}}+\frac{(x+1)(x^2+1)}{x\sqrt{x^2+1}}+\frac{x^2+1}{x}>6$$

$$\frac{x+1}{\sqrt{x^2+1}}+\frac{x+1}{x}\sqrt{x^2+1}+\frac{x^2+1}{x}>6$$

$$\frac{x+1}{\sqrt{x^2+1}}+\frac{x+1}{x}\sqrt{x^2+1}\overset{(M_a\ge M_g)}{\ge}2\sqrt{\frac{x+1}{\sqrt{x^2+1}}\cdot\frac{x+1}{x}\cdot\sqrt{x^2+1}}=\frac{2(x+1)}{\sqrt{x}}\overset{(M_a\ge M_g)}{\ge}\frac{2\cdot 2\sqrt{x}}{\sqrt{x}}=4$$

$$\frac{x^2+1}{x}\ge 2\Leftrightarrow x^2+1>2x\Leftrightarrow (x-1)^2\ge 0$$

$$\Rightarrow \frac{x+1}{\sqrt{x^2+1}}+\frac{x+1}{x}\sqrt{x^2+1}+\frac{x^2+1}{x}>4+2=6\quad \textbf{true}$$

Solution 4 by Tran Hong-Vietnam

$$a^3+b^3\ge ab(a+b)\Rightarrow$$

$$LHS\ge ab(a+b)+\left(\sqrt{a^2+b^2}\right)^3+\frac{4a^2b^2}{a+b+\sqrt{a^2+b^2}}\quad (*)$$

We need to prove: (*) $>4ab\sqrt{a^2+b^2}$

$$\Leftrightarrow \frac{a+b}{\sqrt{a^2+b^2}}+\frac{a^2+b^2}{ab}+\frac{4ab}{\left(a+b+\sqrt{a^2+b^2}\right)\sqrt{a^2+b^2}}>4\quad (1)$$

We have

$$\frac{a+b}{\sqrt{a^2+b^2}}+\frac{a^2+b^2}{ab}+\frac{4ab}{\left(a+b+\sqrt{a^2+b^2}\right)\sqrt{a^2+b^2}}\ge 2\sqrt{\frac{ab}{a^2+b^2}}+\frac{a^2+b^2}{ab}+\frac{4ab}{(\sqrt{2}+1)(a^2+b^2)}\quad (**)$$

Let $f(t)=2t+\frac{1}{t^2}+\frac{4t^2}{\sqrt{2}+1}$ ***with*** $t=\sqrt{\frac{ab}{a^2+b^2}}\ \left(0<t\le\frac{\sqrt{2}}{2}\right)$

$$\Rightarrow f'(t)=2-\frac{2}{t^3}+\frac{8}{1+\sqrt{2}}\cdot t=2\left(\frac{2t^4[\sqrt{2}-1]+t^3-1}{t^3}\right)<0,\forall t\in\left(0,\frac{\sqrt{2}}{2}\right]$$

$$\Rightarrow f(t)\searrow \text{ on }\left(0,\frac{\sqrt{2}}{2}\right]\Rightarrow f(t)\ge f\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}+2+\frac{2}{(\sqrt{2}+1)}=3\sqrt{2}>4$$

$\Rightarrow$ ***(**)*** $> 4 \Rightarrow$ ***(1) true. Proved.***

Solution 5 by Soumava Chakraborty-Kolkata-India

$$a^3+b^3+\left(\sqrt{a^2+b^2}\right)^3+\frac{4a^2b^2}{a+b+\sqrt{a^2+b^2}}\overset{(1)}{>}4ab\sqrt{a^2+b^2}$$

$$(1)\Leftrightarrow (a^3+b^3)\left(a+b+\sqrt{a^2+b^2}\right)+(a^2+b^2)\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)+4a^2b^2>$$

$$>4ab\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)$$

$$\Leftrightarrow 2a^4+2b^4+6a^2b^2-3ab(a^2+b^2)+\sqrt{a^2+b^2}\left(2a^3+2b^3-2ab(a+b)\right)>$$

$$\overset{(2)}{>}ab(a+b)\sqrt{a^2+b^2}$$

$$\because a^3+b^3\ge ab(a+b)\Rightarrow 2a^3+2b^3-2ab(a+b)\ge 0$$

$$\therefore \text{LHS of (2)}\overset{(a)}{>}2a^4+2b^4+6a^2b^2-3ab(a^2+b^2)+\frac{a+b}{2}\left(2a^3+2b^3-2ab(a+b)\right)$$

$$\left(\because \sqrt{a^2+b^2}\ge\frac{a+b}{\sqrt{2}}>\frac{a+b}{2}\right)$$

$$=3a^4+3b^4+4a^2b^2-3ab(a^2+b^2)$$

Also, $\because \sqrt{a^2+b^2}<a+b,\therefore$ ***RHS of (2)***

$$\overset{(b)}{<}ab(a+b)^2$$

(a), (b)⇒ in order to prove (2), it suffices to prove:

$$3a^4+3b^4+4a^2b^2-3ab(a^2+b^2)\ge ab(a+b)^2$$

$$\Leftrightarrow 3a^4+3b^4+2a^2b^2-4ab(a^2+b^2)\ge 0$$

$$\Leftrightarrow 3\{(a^2+b^2)^2-2a^2b^2\}+2a^2b^2-4ab(a^2+b^2)\ge 0$$

$$\Leftrightarrow 3x^2-4xy-4y^2\ge 0 \text{ (where } x=a^2+b^2 \text{ \& } y=ab)$$

$$\Leftrightarrow 3x^2-6xy+2xy-4y^2\ge 0\Leftrightarrow (x-2y)(3x+2y)\ge 0$$

$$\to \text{true} \because a^2+b^2\ge 2ab\Rightarrow x\ge 2y \text{ \& } x,y>0$$

(Proved)

JP.179. In acute ΔABC the following relationship holds:

$$\frac{a\cos A}{b\cos B}+\frac{b\cos B}{c\cos C}+\frac{c\cos C}{a\cos A}\le\frac{3}{8\cos A\cos B\cos C}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a\cos A}{b\cos B}+\frac{b\cos B}{c\cos C}+\frac{c\cos C}{a\cos A}\overset{(1)}{\le}\frac{3}{8\cos A\cos B\cos C}$$

(1) $\Leftrightarrow \frac{(a\cos A)(b\cos B)^2+(b\cos B)(c\cos C)^2+(c\cos C)(a\cos A)^2}{abc\cos A\cos B\cos C}\le\frac{3}{8\cos A\cos B\cos C}$

$$\Leftrightarrow 8\sum(a\cos A)(b\cos B)^2\overset{(2)}{\le}3abc$$

Now, $(a\cos A)(b\cos B)^2=\frac{a(b^2+c^2-a^2)}{2bc}\cdot b^2\cdot\frac{(c^2+a^2-b^2)^2}{4c^2a^2}=\frac{b(b^2+c^2-a^2)(c^2+a^2-b^2)^2}{8c^3a}$

$$\overset{(a)}{=}\frac{a^2b^4(b^2+c^2-a^2)(c^2+a^2-b^2)^2}{8(abc)^3}$$

Similarly, $(b\cos B)(c\cos C)^2\overset{(b)}{=}\frac{b^2c^4(c^2+a^2-b^2)(a^2+b^2-c^2)^2}{8(abc)^3}$ **&**

$$(c\cos C)(a\cos A)^2\overset{(c)}{=}\frac{c^2a^4(a^2+b^2-c^2)(b^2+c^2-a^2)^4}{8(abc)^3}$$

Let $b^2+c^2-a^2=x,\ c^2+a^2-b^2=y,\ a^2+b^2-c^2=z$

Then $\sum a^2=\sum x\Rightarrow a^2=\frac{y+z}{2},\ b^2=\frac{z+x}{2},\ c^2=\frac{x+y}{2}$

Using the above substitution & (a), (b), (c),

(2) becomes: $\left(\frac{y+z}{2}\right)\frac{(z+x)^2}{4}\cdot xy^2+\left(\frac{z+x}{2}\right)\frac{(x+y)^2}{4}\cdot yz^2+\left(\frac{x+y}{2}\right)\frac{(y+z)^2}{4}\cdot zx^2\le$

$$\le 3\left(\frac{y+z}{2}\right)^2\left(\frac{z+x}{2}\right)^2\left(\frac{x+y}{2}\right)^2$$

$$\Leftrightarrow 3(x+y)^2(y+z)^2(z+x)^2\ge 8xy^2(y+z)(z+x)^2+8yz^2(z+x)(x+y)^2+$$
$$+8zx^2(x+y)(y+z)^2$$

$$\Leftrightarrow 3\sum x^4y^2+3\sum x^2y^4+6xyz\left(\sum x^3\right)+2xyz\left(\sum x^2y\right)$$
$$\overset{(3)}{\ge}2\sum x^3y^3+6xyz\left(\sum xy^2\right)+18x^2y^2z^2$$

It should be noted that, $\because (b^2+c^2-a^2)$ **etc** >0

($\because \Delta ABC$ is acute-angled), $\therefore x, y, z > 0$

Now, $x^2 + z^2x \overset{A-G}{\underset{(i)}{\geq}} 2zx^2, y^3 + x^2y \overset{A-G}{\underset{(ii)}{\geq}} 2xy^2$ *&* $z^3 + y^2z \overset{A-G}{\underset{(iii)}{\geq}} 2yz^2$

(i)+(ii) +(iii)$\Rightarrow \sum x^3 + \sum x^2y \geq 2\sum xy^2 \Rightarrow 3xyz(\sum x^3 + \sum x^2y) \overset{(iv)}{\geq} 6xyz(\sum xy^2)$

Also, $\sum x^2y^4 \overset{(v)}{\geq} xy^2 \cdot yz^2 + yz^2 \cdot zx^2 + zx^2 \cdot xy^2 = xyz(\sum x^2y)$

Again, $\sum x^4y^2 + \sum x^2y^4 \overset{A-G}{\underset{(vi)}{\geq}} 2\sum x^3y^3$

Lastly, $2\sum x^4y^2 + \sum x^2y^4 + 3xyz(\sum x^3) \overset{A-G}{\underset{(vii)}{\geq}} 2 \cdot (3x^2y^2z^2) + (3x^2y^2z^2) + 3xyz \cdot 3xyz = 18x^2y^2z^2$

(iv)+(v) +(vi) +(vii)⇒(3) is true

(Hence proved)

Solution 2 by Marian Ursărescu-Romania

We use the orthic triangle: Because ΔABC is acute let $a' = a\cos A$, $b' = b\cos B$, $c' = c\cos C$ the sides of the orthic triangle of ABC: but $R' = \frac{R}{2}$, R' = circumradii of orthic ΔABC, $r' = 2R\cos A\cos B\cos C$, r' = inradius $\Rightarrow r' = 4R'\cos A\cos B\cos C \Rightarrow$

$\Rightarrow \cos A\cos B\cos C = \frac{r'}{4R'} \Rightarrow$ we must show this: $\frac{a'}{b'} + \frac{b'}{c'} + \frac{c'}{a'} \leq \frac{3R'}{2r'}$, which means we must show $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq \frac{3R}{2r}$ for any Δ (1)

$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \leq (a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$ *(2) (from Cauchy)*

But $a^2 + b^2 + c^2 \leq 9R^2$ *and* $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}$ *(3)*

From (2)+(3)$\Rightarrow \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \leq \frac{9R^2}{4r^2} \Leftrightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq \frac{3R}{2r} \Rightarrow$ *(1) it is true.*

JP.180. If $a, b \geq 0$ then:

$$\begin{cases} 4ab \leq \sqrt{a^2 + b^2}\left(a + b + \sqrt{a^2 + b^2}\right) \\ 4ab\sqrt{a^2 + b^2} \leq (a^2 + b^2)\left(a + b + \sqrt{a^2 + b^2}\right) \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$\text{Let } f(t)=5t^2-8t+5$$

$$D=64-100=-36<0 \because D<0\Rightarrow F(t)>0$$

$$\text{Put } t=\frac{a}{b}>0,\{a,b>0\}\Rightarrow F(t)=5t^2-8t+5>0$$

$$\text{Putting } t=\left(\frac{a}{b}\right)$$

$$5\left(\frac{a^2}{b^2}\right)-8\left(\frac{a}{b}\right)+5>0$$

$$5a^2-8ab+5b^2>0\Rightarrow 4a^2+4b^2+8ab<9a^2+9b^2$$

$$\Rightarrow 4(a+b)^2<9(a^2+b^2)\Rightarrow 2(a+b)<3\sqrt{a^2+b^2}$$

$$\Rightarrow 2\left(a+b-\sqrt{a^2+b^2}\right)<\sqrt{a^2+b^2}$$

$$\Rightarrow\frac{2\left(a+b-\sqrt{a^2+b^2}\right)\left(a+b+\sqrt{a^2+b^2}\right)}{\left(a+b+\sqrt{a^2+b^2}\right)}<\sqrt{a^2+b^2}$$

$$\Rightarrow 2(2ab)<\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)$$

Also, if $a=b=0$, equality holds $\Rightarrow 4ab\le\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)$

Proved

Multiplying both sides by $\sqrt{a^2+b^2}$

$$4ab\sqrt{a^2+b^2}\le(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)$$

Proved

Solution 2 by Khaled Abd Almuty-Damascus-Syria

If $a,b\ge 0$ then:

1) $4ab\le\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)$

2) $4ab\sqrt{a^2+b^2}\le(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)$

1) We know: $\sqrt{x+y}\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}\right)\ge 2\sqrt{2},\forall x,y\in\mathbb{R}_+^*$

For $x=a^2,y=b^2$: $\sqrt{a^2+b^2}\left(\frac{1}{a}+\frac{1}{b}\right)\ge 2\sqrt{2}$

$$\sqrt{a^2+b^2}\cdot\left(\frac{b+a}{ab}\right)\ge 2\sqrt{2}\Rightarrow(a+b)\sqrt{a^2+b^2}\ge 2\sqrt{2}ab$$

$$(a+b)\sqrt{a^2+b^2}+a^2+b^2\ge a^2+b^2+2\sqrt{2}ab \quad (*)$$

Let us prove that $a^2+b^2+2\sqrt{2}ab\overset{?}{\ge}4ab$

$$\frac{a^2}{ab}+\frac{b^2}{ab}+2\sqrt{2}\overset{?}{\ge}4,\frac{a}{b}+\frac{b}{a}+2\sqrt{2}\overset{?}{\ge}4\ \left\{\frac{a}{b}=x,\frac{b}{a}=\frac{1}{x}\right\}$$

In order to prove that: let $f(x)=x+\frac{1}{x}, D=\,]0,+\infty[$

$$'f(x)=1-\frac{1}{x^2}=\frac{x^2-1}{x^2},\ 'f(x)=0\Rightarrow x=1, f(1)=\frac{3}{2}$$

x	0		1		$+\infty$
$f'(x)$		$- - - - - - - - - - - - -$	0	$+ + + + + + + + + + + + + + + +$	
$f(x)$		$+\infty$ ↘	$\frac{3}{2}$	↗	$+\infty$

$$\forall x\in\,]0,+\infty[: f(x)\ge\frac{3}{2}\Rightarrow x+\frac{1}{x}\ge\frac{3}{2}\ge 4-2\sqrt{2}$$

So: $\frac{a}{b}+\frac{b}{a}\ge 4-2\sqrt{2}\Rightarrow\frac{a}{b}+\frac{b}{a}+2\sqrt{2}\ge 4;\ \ a\cdot b>0$

$$a^2+b^2+2\sqrt{2}ab\ge 4ab$$

From relation (*): $(a+b)\sqrt{a^2+b^2}+a^2+b^2\ge a^2+b^2+2\sqrt{2}ab\ge 4ab$

So: $(a+b)\sqrt{a^2+b^2}+\sqrt{a^2+b^2}\cdot\sqrt{a^2+b^2}\ge 4ab$

$$\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)\ge 4ab$$

2) From relation 1):

$$4ab\le\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right); x\sqrt{a^2+b^2}$$

$$4ab\sqrt{a^2+b^2}\le(a^2+b^2)\left(a+b+\sqrt{a^3+b^3}\right)$$

Note if $a=0$ ***and*** $b=0$ ***the relation (1) it is true.***

And if $a=0$ ***and*** $b\ne 0$***:*** $0\le a(2a); 2a^2\ge 0$ ***it is true***

Solution 3 by Michael Sterghiou-Greece

Both inequalities are homogeneous so we can assume $a^2+b^2=1$

The both become: $4ab\le a+b+1 \quad$ ***(2)***

Now, $a^2+b^2=1\to a\le 1\wedge b\le 1\to a^2\le a\wedge b^2\to a+b\ge 1$

Also $a^2+b^2=1\ge 2ab\to ab\le\frac{1}{2}$

(2) $\rightarrow 4ab \leq 2 \leq a+b+1$ **which is true.**

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\forall a,b \geq 0, 4ab \overset{(1)}{\leq} \sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right) \text{ \& }$$

$$4ab\sqrt{a^2+b^2} \overset{(2)}{\leq} (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)$$

$$\textbf{(1)} \Leftrightarrow 4ab \leq (a+b)\sqrt{a^2+b^2}+a^2+b^2 \Leftrightarrow (a-b)^2+(a+b)\sqrt{a^2+b^2} \overset{(1a)}{\geq} 2ab$$

$$\because \left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0, \therefore a+b \overset{(a)}{\geq} 2\sqrt{ab}$$

$$\because a^2+b^2 \geq 2ab(as\ (a-b)^2 \geq 0),$$

$$\therefore \sqrt{a^2+b^2} \overset{(b)}{\geq} \sqrt{2ab}(\because a,b \geq 0)$$

$$\textbf{(a).(b)} \Rightarrow (a+b)\sqrt{a^2+b^2} \geq 2\sqrt{2}ab$$

$$(\because a+b \geq 0 \text{ as } a,b \geq 0 \text{ \& } \sqrt{ab} \geq 0 \text{ as } a,b \geq 0)$$

$$\overset{?}{\geq} 2ab \Leftrightarrow 2ab\left(\sqrt{2}-1\right) \overset{?}{\geq} 0 \rightarrow \textbf{true}$$

$$\because ab \geq 0(\because a,b \geq 0) \text{ \& } \sqrt{2}-1 > 0$$

$\therefore$ **(1) is proved**

$$\textbf{(2)} \Leftrightarrow (a^2+b^2)(a+b)+(a^2+b^2)\sqrt{a^2+b^2} \overset{(2a)}{\geq} 4ab\sqrt{a^2+b^2}$$

Now, $(a^2+b^2)(a+b) = \sqrt{a^2+b^2}\sqrt{a^2+b^2}(a+b)$

$$\overset{(c)}{\geq} \sqrt{a^2+b^2}\sqrt{2ab}\left(2\sqrt{ab}\right)$$

$$\left(\because a^2+b^2 \geq 2ab \text{ as } (a-b)^2 \geq 0 \text{ \&} a+b \geq 2\sqrt{ab} \text{ as} \left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0\right)$$

$$= 2\sqrt{2}ab\sqrt{a^2+b^2}$$

Also, $(a^2+b^2)\sqrt{a^2+b^2} \overset{(d)}{\geq} 2ab\sqrt{a^2+b^2}(\because a^2+b^2 \geq 2ab)$

(c) + **(d)** $\Rightarrow$ **RHS of (2)** $\geq 2ab\sqrt{a^2+b^2}\left(1+\sqrt{2}\right)$

$$\geq 4ab\sqrt{a^2+b^2}\left(\because 1+\sqrt{2} > 2 \text{ \&} ab\sqrt{a^2+b^2} \geq 0 \text{ as } a,b \geq 0\right)$$

$\Rightarrow$ **(2) is true. (Done).**

Solution 5 by Tran Hong-Vietnam

$$\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)\geq 4ab \quad \textbf{(1)}$$

We have: $\sqrt{a^2+b^2}\geq \frac{a+b}{\sqrt{2}}$

$$LHS_{(1)}\geq \frac{a+b}{\sqrt{2}}\left(a+b+\frac{a+b}{\sqrt{2}}\right)=\left(\frac{1+\sqrt{2}}{2}\right)(a+b)^2$$

$\geq \frac{1+\sqrt{2}}{2}\cdot 4ab\geq 4ab\Rightarrow$ ***(1) true, equality*** $\Leftrightarrow a=b=0.$

Using (1) we have

$$\Leftrightarrow \sqrt{a^2+b^2}\cdot\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)\geq 4ab\sqrt{a^2+b^2}$$

$$\Leftrightarrow (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)\geq 4ab\sqrt{a^2+b^2}$$

Proved.

SP.166. Let $n\in\mathbb{N}^*$ and $a_k\in\mathbb{R},\forall k=\overline{1;n}$. Find:

$$\Omega=\int \ln\left(\prod_{k=1}^{n}(x-a_k)\right)dx$$

$$(x>\max\{a_k|\forall k=\overline{1,n}\})$$

Proposed by Nguyen Van Nho –Nghe An – Vietnam

Solution by Tran Hong-Vietnam

$$\Omega=\int\left(\prod_{k=1}^{n}(x-a_k)\right)dx=\sum_{k=1}^{n}\int \ln(x-a_k)\,dx$$

Let $I=\int \ln(x-a_k)\,dx.\ \ (x>\max\{a_k|k=1,2,\dots,n\})$

$$=x\ln(x-a_k)-\int\frac{x}{x-a_k}dx=x\ln(x-a_k)-\int\left(1+\frac{a_k}{x-a_k}\right)dx$$

$$=x\ln(x-a_k)-x-a_k\ln(x-a_k)+C$$

$$=(x-a_k)\ln(x-a_k)-x+C\ \ \textit{(C: const)}$$

$$\Rightarrow \Omega=-nx+\sum_{k=1}^{n}(x-a_k)\ln(x-a_k)+D;$$

(D: const)

SP.167. Let x,y,z be positive real numbers such that: $xyz=1$. Prove that:

$$\frac{x}{\sqrt{2(x^4+y^4)}+4xy}+\frac{y}{\sqrt{2(y^4+z^4)}+4yz}+\frac{z}{\sqrt{2(z^4+x^4)}+4zx}+\frac{2(x+y+z)}{3}\geq\frac{5}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Bao Truong-Vietnam

$$\sum\frac{a}{\sqrt{2(a^4+b^4)}+4ab}\geq\sum\frac{a}{2(a^2+b^2+ab)}=\sum\frac{a}{2\left[\frac{3}{2}(a+b)^2-\frac{1}{2}(a-b)^2-3ab\right]}\geq$$

$$\geq\sum\frac{a}{3(a+b)^2-6ab}\Rightarrow\sum\frac{a}{\sqrt{2(a^4+b^4)}+4ab}+\sum\frac{3(a+b)^2-6ab}{9(a+b)^2}\geq\frac{2}{3}\sum\frac{\sqrt{a}}{a+b}\geq$$

$$\geq\frac{2}{\sqrt[3]{\prod(a+b)}}\geq\frac{3}{(a+b+c)}\Rightarrow\sum\frac{a}{\sqrt{2(a^4+b^4)}+4ab}+\frac{2}{3}\sum a\geq$$

$$\geq\frac{3}{\sum a}+\frac{2}{3}\sum\frac{ab}{(a+b)^2}-1+\frac{2}{3}\sum a\geq\frac{3}{\sum a}+\frac{2}{\sqrt[3]{\prod(a+b)^2}}+\frac{2}{3}\sum a-1$$

$$\sum\frac{a}{\sqrt{2(a^4+b^4)}+4ab}+\frac{2}{3}\sum a\geq\frac{3}{\sum a}+\frac{9}{2(\sum a)^2}+\frac{2}{3}\sum a-1=\frac{3}{\sum a}+\frac{\sum a}{3}+$$

$$+\frac{9}{2(\sum a)^2}+\frac{\sum a}{6}+\frac{\sum a}{6}-1\Rightarrow\sum\frac{a}{\sqrt{2(a^4+b^4)}+4ab}+\frac{2}{3}\sum a\geq 2+\frac{3}{2}-1=\frac{5}{2}\ \text{(R.H.D.)}$$

Solution 2 by Michael Sterghiou-Greece

$$x,y,z>0\wedge xyz=1\to\sum_{cyc}\frac{x}{\sqrt{2(x^4+y^4)}+4xy}+\frac{2\sum_{cyc}x}{3}\geq\frac{5}{2}\quad (1)$$

Let $(p,q,r)=\left(\sum_{cyc}x,\sum_{cyc}xy,xyz\right)$. $r=1, p\geq 3, q\geq 3$. *(AM-GM)*

Because $\left[\sqrt{2(x^4+y^4)}\right]^2-4(x^2+y^2-xy)^2=-2(x-y)^4\leq 0$, *(1) can be written as*

$\sum_{cyc}\frac{(\sqrt{x})^2}{2(x^2+y^2+xy)}+\frac{2p}{3}-\frac{5}{2}\geq 0$. *Using BCS we need to show that:* $\frac{9}{(2p^2-3q)}+\frac{4p}{3}-5\geq 0$.

(2) because $\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2\geq 3^2$ *and* $\sum_{cyc}(x^2+y^2+xy)=2p^2-3q$. *(2) reduces to*

$(4p-15)(2p^2-3q)+27\geq 0$. *(3). As $p^2-3q\geq 0$ if $4p-15\geq 0$ we are done. If*

$4p-15<0$ *then (3) reduces to:* $8p^3-30p^2+3q(15-4p)+27$ *which must be* ≥ 0*. We know that* $q^2\geq 3pr=3p$ *and as* $15-4p>0$ *it suffices to show that*

$$8p^3-30p^2+3\sqrt{3}p(15-4p)+27=f(p)\geq 0, f(3)=0, f'(p)=\frac{3}{2\sqrt{p}}\cdot g(p),$$

$$g(p)=16p^{\frac{5}{2}}-40p^{\frac{3}{2}}-12\sqrt{3}p+15\sqrt{3}; g'(p)=40p^{\frac{3}{2}}-60p^{\frac{1}{2}}+12\sqrt{3}$$ *and*

$g''(p)=\frac{30(2p-1)}{\sqrt{p}}\geq 0$ *so easily we can deduce that* $g(p)>0\to f'(p)>0\to$

$f(p)\uparrow\to f(p)>f(3)=0$*. Done!*

SP.168. Let x,y,z be positive real numbers.

Find the minimum possible value of:

$$\frac{x}{y+z}+\frac{y}{z+x}+2\sqrt{\frac{1}{2}+\frac{z}{x+y}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kelvin Kong-Malaysia

I will prove that $A=\frac{x}{y+z}+\frac{y}{z+x}+2\sqrt{\frac{1}{2}+\frac{z}{x+y}}\geq 3$

Let $B=\frac{x}{y+z}+\frac{y}{z+x}, C=2\sqrt{\frac{1}{2}+\frac{z}{x+y}}$

$$C=\sqrt{\frac{2}{x+y}}\sqrt{(y+z)+(z+x)}\geq\sqrt{\frac{2}{x+y}}\sqrt{2\sqrt{(y+z)(z+x)}}=\frac{2\sqrt[4]{(y+z)(z+x)}}{\sqrt{x+y}}$$

By using QM-AM inequality: $\sqrt{\frac{x^2+y^2}{2}}\geq\frac{x+y}{2}$*, we have* $x^2+y^2\geq\frac{1}{2}(x+y)^2$

$$B=\frac{x^2+y^2+xz+yz}{(y+z)(z+x)}\geq\frac{\frac{1}{2}(x+y)^2+(x+y)z}{(y+z)(z+x)}=\frac{(x+y)[(y+z)+(z+x)]}{2(y+z)(z+x)}$$

$$B\geq\frac{(x+y)\cdot 2\sqrt{(y+z)(z+x)}}{2(y+z)(z+x)}=\frac{x+y}{\sqrt{(y+z)(z+x)}}$$

Therefore

$$A = B + C \geq \frac{x+y}{\sqrt{(y+z)(z+x)}} + \frac{\sqrt[4]{(y+z)(z+x)}}{\sqrt{x+y}} + \frac{\sqrt[4]{(y+z)(z+x)}}{\sqrt{x+y}}$$

$$\geq 3\sqrt[3]{\frac{x+y}{\sqrt{(y+z)(z+x)}} \cdot \frac{\sqrt[4]{(y+z)(z+x)}}{\sqrt{x+y}} \cdot \frac{\sqrt[4]{(y+z)(z+x)}}{\sqrt{x+y}}} = 3$$

In conclusion: $A = \frac{x}{y+z} + \frac{y}{z+x} + 2\sqrt{\frac{1}{2} + \frac{z}{x+y}} \geq 3$ ***where equality holds when*** $x = y = z$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{y+z} + \frac{y}{z+x} = \frac{x(z+x) + y(y+z)}{(y+z)(z+x)} \underset{(1)}{\overset{A-G}{\geq}} \frac{4\{x^2 + y^2 + z(x+y)\}}{(2z+x+y)^2} \overset{Chebyshev}{\geq} \frac{4\left[\frac{1}{2}(x+y)^2 + z(x+y)\right]}{(2z+x+y)^2}$$

$$= \frac{2(x+y)(2z+x+y)}{(2z+x+y)^2} = \frac{2(x+y)}{2z+x+y}$$

$$(1) \Rightarrow \frac{x}{y+z} + \frac{y}{z+x} + 2\sqrt{\frac{1}{2} + \frac{z}{x+y}} \geq \frac{2(x+y)}{2z+x+y} + \sqrt{\frac{2z+x+y}{2(x+y)}} + \sqrt{\frac{2z+x+y}{2(x+y)}}$$

$$\overset{A-G}{\geq} 3\sqrt[3]{\frac{2(x+y)(2z+x+y)}{2(x+y)(2z+x+y)}} = 3$$

$\Rightarrow$ ***regd. min value*** $= 3$, ***which occurs at*** $x = y = z$.

Solution 3 by Tran Hong-Vietnam

$$P = \frac{x}{y+z} + \frac{y}{z+x} + 2\sqrt{\frac{1}{2} + \frac{z}{x+y}} = \frac{x^2}{xy+xz} + \frac{y^2}{yz+yx} + 2\sqrt{\frac{x+y+2z}{2(x+y)}}$$

$$\overset{(Schwarz)}{\geq} \frac{(x+y)^2}{2xy + z(x+y)} + 2\sqrt{\frac{x+y+2z}{2(x+y)}}$$

$$\geq \frac{2(x+y)^2}{(x+y)^2 + 2z(x+y)} + \sqrt{\frac{x+y+2z}{2(x+y)}} + \sqrt{\frac{x+y+2z}{2(x+y)}}$$

$$= \frac{2(x+y)}{x+y+2z} + \sqrt{\frac{x+y+2z}{2(x+y)}} + \sqrt{\frac{x+y+2z}{2(x+y)}}$$

$$\overset{(Schwarz)}{\geq} 3\sqrt[3]{\frac{2(x+y)}{x+y+2z} \cdot \frac{x+y+2z}{2(x+y)}} = 3$$

$$\Rightarrow P_{\min} = 3 \Leftrightarrow x = y = z$$

SP.169. Prove that for all non-negative real numbers a, b, c

$$\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{c+a+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \geq 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

Suppose: $a+b+c=3 \Rightarrow 0 < a, b, c < 3$

Inequality $\Leftrightarrow \sqrt{\frac{a^2+2}{4-a}} + \sqrt{\frac{b^2+2}{4-b}} + \sqrt{\frac{c^2+2}{4-c}} \geq 3.$ ***(1)***

For all $0 < x < 3$ ***we have:*** $\sqrt{\frac{x^2+2}{4-x}} \geq \frac{1}{2}(x+1)$ ***(*)***

$\Leftrightarrow \frac{x^2+2}{4-x} \geq \frac{1}{2}(x+1)^2 \Leftrightarrow \frac{(x-1)^2(x+4)}{4-x} \geq 0$ ***. (True because*** $0 < x < 3$***)***

Using (*) with $0 < a, b, c < 3$ ***we have***

$$\sqrt{\frac{a^2+2}{4-a}} + \sqrt{\frac{b^2+2}{4-b}} + \sqrt{\frac{c^2+2}{4-c}} \geq \frac{1}{2}(a+1) + \frac{1}{2}(b+1) + \frac{1}{2}(c+1)$$

$= \frac{1}{2}(a+b+c+3) = \frac{6}{2} = 3$***. Proved. Equality*** $\Leftrightarrow a=b=c=1.$

Solution 2 by Tran Hong-Vietnam

$$LHS = \frac{\sqrt{(a^2+2)(2+1)}}{\sqrt{(b+c+1)(2+1)}} + \sqrt{\frac{(b^2+2)(2+1)}{(c+a+1)(2+1)}} + \sqrt{\frac{(c^2+2)(2+1)}{(a+b+1)(2+1)}}$$

$$\geq \frac{\sqrt{(a+2)^2}}{\frac{b+c+1+3}{2}} + \frac{\sqrt{(b+2)^2}}{\frac{c+a+1+3}{2}} + \frac{\sqrt{(c+2)^2}}{\frac{a+b+1+3}{2}} = \frac{2a+4}{b+c+4} + \frac{2b+4}{c+a+4} + \frac{2c+4}{a+b+4}$$

$$= \frac{2a+4}{b+c+4} + 2 + \frac{2b+4}{c+a+4} + 2 + \frac{2c+4}{a+b+4} + 2 + 6$$

$$= 2(a+b+c+6)\left(\frac{1^2}{b+c+4} + \frac{1^2}{c+a+4} + \frac{1^2}{a+b+4}\right) - 6$$

$$\geq 2(a+b+c+6)\cdot\frac{(1+1+1)^2}{2(a+b+c+6)}-6=9-6=3$$

Proved. Equality $\Leftrightarrow a=b=c$.

Solution 3 by Remus Florin Stanca-Romania

We know that for any real numbers $x,y,z>0$ ***we have that*** $\sqrt{\frac{x^2+y^2+z^2}{3}}\geq$

$$\geq\frac{x+y+z}{3}\Leftrightarrow\sqrt{\frac{a^2+1+1}{3}}\geq\frac{a+2}{3}\Rightarrow\sqrt{a^2+2}\geq\frac{a+2}{\sqrt{3}}\Rightarrow$$

$$\succ\sqrt{\frac{a^2+2}{b+c+1}}\geq\frac{a+2}{\sqrt{3(b+c+1)}}$$

$$\sqrt{\frac{b^2+2}{a+c+1}}\geq\frac{b+2}{\sqrt{3(a+c+1)}}$$

$$\sqrt{\frac{c^2+2}{a+b+1}}\geq\frac{c+2}{\sqrt{3(a+b+1)}}$$

------------------------------------- +

$$\sqrt{\frac{a^2+2}{b+c+1}}+\sqrt{\frac{b^2+2}{a+c+1}}+\sqrt{\frac{c^2+2}{a+b+1}}\geq\sum\frac{a+2}{\sqrt{3(b+c+1)}}\quad \textbf{\textit{(1)}}$$

$$\sqrt{3(b+c+1)}\leq\frac{b+c+4}{2}\succ\frac{a+2}{\sqrt{3(b+c+1)}}\geq\frac{2a+4}{b+c+4}\succ$$

$$\Rightarrow\sum\frac{a+2}{\sqrt{3(b+c+1)}}\geq 2\sum\frac{a+2}{b+2+c+2}$$ ***(2), we know also, that***

$\sum\frac{x}{y+z}\geq\frac{3}{2}$, ***we put*** $x=a+2$, $y=b+2$, $z=c+2\succ 2\sum\frac{a+2}{b+2+c+2}\geq 3$ ***(3)***

$$(1)\overset{\succ}{(2)}(3)\sqrt{\frac{a^2+2}{b+c+1}}+\sqrt{\frac{b^2+2}{a+c+1}}+\sqrt{\frac{c^2+2}{a+b+1}}\geq 3.$$ ***(Q.E.D.)***

Solution 4 by Soumava Chakraborty-Kolkata-India

$\because a,b,c\geq 0,\frac{a^2+2}{b+c+1}$, ***etc*** >0

$\therefore$ ***by A-G, LHS*** $\geq 3\sqrt[3]{\sqrt{\frac{(a^2+2)(b^2+2)(c^2+2)}{(b+c+1)(c+a+1)(a+b+1)}}}\overset{?}{\geq}3$

$$\Leftrightarrow (a^2+2)(b^2+2)(c^2+2) \overset{?}{\geq} (b+c+1)(c+a+1)(a+b+1)$$

$$\Leftrightarrow a^2b^2c^2 + 2\sum a^2b^2 + 3\sum a^2 + 7 \underset{(1)}{\overset{?}{\geq}} \sum a^2b + \sum ab^2 + 2abc + 3\sum ab + 2\sum a$$

Now, $\frac{1}{2}a^2(b-1)^2 \geq 0 \Rightarrow \frac{1}{2}(a^2b^2+a^2) \overset{(i)}{\geq} a^2b$

$$\frac{1}{2}b^2(c-1)^2 \geq 0 \Rightarrow \frac{1}{2}(b^2c^2+b^2) \overset{(ii)}{\geq} b^2c$$

$$\frac{1}{2}c^2(a-1)^2 \geq 0 \Rightarrow \frac{1}{2}(c^2a^2+c^2) \overset{(iii)}{\geq} c^2a$$

$$\frac{1}{2}b^2(a-1)^2 \geq 0 \Rightarrow \frac{1}{2}(a^2b^2+b^2) \overset{(iv)}{\geq} ab^2$$

$$\frac{1}{2}c^2(b-1)^2 \geq 0 \Rightarrow \frac{1}{2}(b^2c^2+c^2) \overset{(v)}{\geq} bc^2$$

$$\frac{1}{2}a^2(c-1)^2 \geq 0 \Rightarrow \frac{1}{2}(c^2a^2+a^2) \overset{(vi)}{\geq} ca^2$$

Also, $\because \frac{1}{2}\left[\sum(a-b)^2\right] \geq 0, \therefore \sum a^2 \overset{(vii)}{\geq} \sum ab$

$$\because \sum(a-1)^2 \geq 0, \therefore \sum a^2 + 3 \overset{(viii)}{\geq} 2\sum a$$

$$\because (abc-1)^2 \geq 0, \therefore a^2b^2c^2 + 1 \overset{(ix)}{\geq} 2abc$$

$$\because \sum(ab-1)^2 \geq 0, \therefore \sum a^2b^2 + 3 \overset{(x)}{\geq} 2\sum ab$$

(i)+(ii)+(iii)+(iv)+(v)+$(vi)+(vii)+(viii)+(ix)+(x) \Rightarrow$ ***(1) is true (proved)***

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c \geq 0$***, we have***

$$(a^2+1+1)(b^2+1+1) \geq (a+b+1)^2$$

$$(b^2+1+1)(c^2+1+1) \geq (b+c+1)^2$$

$$(c^2+1+1)(a^2+1+1) \geq (c+a+1)^2$$

$$\Rightarrow (a^2+2)(b^2+2)(c^2+2) \geq (a+b+1)(b+c+1)(c+a+1)$$

Hence $\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{c+a+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \geq 3\sqrt[3]{\sqrt[2]{\left(\frac{a^2+2}{b+c+1}\right)\left(\frac{b^2+2}{c+a+1}\right)\left(\frac{c^2+2}{a+b+1}\right)}} \geq 3$

Iff $\sqrt[6]{\left(\frac{a^2+2}{b+c+1}\right)\left(\frac{b^2+2}{c+a+1}\right)\left(\frac{c^2+2}{a+b+1}\right)} \geq 1$ *and it is to be true.*

Therefore, it is to be true

Solution 6 by Soumitra Mandal-Chandar Nagore-India

By Cauchy-Schwarz inequality,

$$(a^2+1+1)(b^2+1+1) \geq \left(\sqrt{a^2\cdot 1}+\sqrt{b^2\cdot 1}+\sqrt{1\cdot 1}\right)^2 = (a+b+1)^2$$

Similarly, $(b^2+2)(c^2+2) \geq (b+c+1)^2$ *and* $(c^2+2)(a^2+2) \geq (c+a+1)^2$

Multiplying the above we have $\prod_{cyc}(a^2+2) \geq \prod_{cyc}(a+b+1)$

$$\sum_{cyc}\sqrt{\frac{a^2+2}{b+c+1}} \overset{AM\geq GM}{\underset{(3)}{\geq}} 3\sqrt[3]{\prod_{cyc}\sqrt{\frac{a^2+2}{b+c+1}}} = 3$$

SP.170. Let a, b, c, d be positive real numbers such that $a+b+c+d=2$. Prove that:

$$\frac{a}{\sqrt{b+\sqrt[3]{cda}}}+\frac{b}{\sqrt{c+\sqrt[3]{dab}}}+\frac{c}{\sqrt{d+\sqrt[3]{abc}}}+\frac{d}{\sqrt{a+\sqrt[3]{bcd}}} \geq 2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

Using Cauchy's inequality, we have:

$$LHS \geq \frac{a}{\sqrt{b+\frac{c+d+a}{3}}}+\frac{b}{\sqrt{c+\frac{a+b+d}{3}}}+\frac{c}{\sqrt{d+\frac{a+b+c}{3}}}+\frac{d}{\sqrt{a+\frac{b+c+d}{3}}} =$$

$$\sqrt{\frac{3}{2}}\left(\frac{a}{\sqrt{1+b}}+\frac{b}{\sqrt{1+c}}+\frac{c}{\sqrt{1+d}}+\frac{d}{\sqrt{1+a}}\right) \quad (*)$$

Let $f(x)=\frac{1}{\sqrt{1+x}}\ (0<x<2) \Rightarrow f''(x)=\frac{3}{4(1+x)^{\frac{5}{2}}} > 0\ (\forall x \in (0,2))$;

Using Jensen's inequality:

$$2\cdot\sqrt{\frac{3}{2}}\left[\frac{a}{b}f(b)+\frac{b}{2}f(c)+\frac{c}{2}f(d)+\frac{d}{2}f(a)\right] \geq 2\sqrt{\frac{3}{2}}f\left(\frac{ab+bc+cd+da}{2}\right) =$$

$$= 2\sqrt{\frac{3}{2}} \cdot \frac{1}{\sqrt{1+\frac{ab+bc+cd+da}{2}}} = \sqrt{12} \cdot \frac{1}{\sqrt{2+ab+bc+cd+da}} \quad (**)$$

Because:

$$4 - 4(ab+bc+cd+da) = (a+b+c+d)^2 - 4(ab+bc+cd+da) =$$

$$= (a-b+c-d)^2 \geq 0 \rightarrow ab+bc+cd+da \leq 1;$$

$$\Rightarrow (**) \geq \sqrt{12} \cdot \sqrt{\frac{1}{2+1}} = 2. \textit{ Proved. Equality } \Leftrightarrow a = b = c = d = \frac{1}{2}.$$

Solution 2 by Minh Tam Le-Vietnam

$$\sum_{cyc}^{a,b,c,d} \frac{a}{\sqrt{b + \sqrt[3]{acd}}} \overset{CBS}{\geq} \frac{(\sum a)^2}{\sum_{cyc}^{a,b,c,d} a\sqrt[3]{b + \sqrt{acd}}} \overset{AM-GM}{\geq} \sum_{cyc}^{a,b,c,d} \frac{4}{a\sqrt{b + \frac{a+c+d}{3}}}$$

$$= \frac{4}{\sum_{cyc}^{a,b,c,d} a\sqrt{\frac{2+2b}{3}}} = \frac{4}{\sum_{cyc}^{a,b,c,d} \frac{a}{3}\sqrt{3(2+2b)}} \overset{AM-GM}{\geq} \frac{4}{\sum_{cyc}^{a,b,c,d} \frac{a}{3}\left(\frac{3+2+2b}{2}\right)} =$$

$$= \frac{4}{\frac{5\sum_{cyc}^{a,b,c,d} a}{6} + \sum_{cyc}^{a,b,c,d} \frac{ab}{3}} = \frac{4}{\frac{5}{3} + \sum_{cyc}^{a,b,c,d} \frac{ab}{3}} \geq 2 \quad (*)$$

$$(*)\ 2\sum_{cyc}^{a,b,c,d} ab = \left(\sum_{cyc}^{a,b,c,d} a\right)^2 - \sum_{cyc}^{a,b,c,d} a^2 - 2ac + 2bd = 4 - [(a+c)^2 + (b+d)^2]$$

$$\overset{CBS}{\leq} 4 - \frac{1}{2}\left(\sum_{cyc}^{a,b,c,d} a\right)^2 = 2 \Leftrightarrow \sum_{cyc}^{a,b,c,d} ab \leq 1$$

SP.171. Let a, b, c be positive real numbers such that: $abc = 1$. Find the minimum value of:

$$P = \frac{a^4}{b^5\sqrt{5(a^4+4)}} + \frac{b^4}{c^5\sqrt{5(b^4+4)}} + \frac{c^4}{a^5\sqrt{5(c^4+4)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $abc = 1, a, b, c > 0$ ***we have***

$$\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^4 \ge \left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+2\left(\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}\right)\right)^2$$

$$\ge 9\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+2\left(\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}\right)\right)$$

Consider $\frac{a^4}{b^5\sqrt{5(a^4+4)}}+\frac{b^4}{c^4\sqrt{5(b^4+4)}}+\frac{c^4}{a^5\sqrt{5(c^4+4)}}$

$$\ge \frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{\sqrt{5(a^4b^2+4b^2)}+\sqrt{5(b^4c^2+4c^2)}+\sqrt{5(c^4a^2+4c^2)}}$$

$$\ge \frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{\sqrt{15\left(a^4b^2+b^4c^2+c^4a^2+4(a^2+b^2+c^2)\right)}} \ge \frac{3}{5}$$

If $\frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{15\left(a^4b^2+b^4c^2+c^4a^2+4(a^2+b^2+c^2)\right)} \ge \frac{9}{25}$

If $\frac{9\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+2\left(\frac{a^2}{b^2}+\frac{c^2}{b^2}+\frac{b^2}{c^2}\right)\right)}{15\left(a^4b^2+b^4c^2+c^4a^2+4(a^2+b^2+c^2)\right)} \ge \frac{9}{25}$

If $\frac{\frac{a^4}{b^5}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+2\left(\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{c^2}\right)}{a^4b^2+b^4c^2+c^4a^2+4(a^2+b^2+c^2)} \ge \frac{3}{5}$ ***ok***

and if is to be true, because

$$5\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}\right)+7(a^4b^2+b^4c^2+c^4a^2)=$$

$$=5(a^8c^4+b^8a^4+c^8b^4)+7(a^4b^2+b^4a^2+c^4a^2)$$

$$\ge 10(a^5c+c^5b+b^5a)+2\left(\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}\right)=10\left(\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\right)+2\left(\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}\right)$$

$$\ge 6\left(\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\right)+6(a^2+b^2+c^2):\frac{a^4}{b}+\frac{a^4}{b}+\frac{b^2}{a^2}\ge 3a^2$$

$$\ge 12(a^2+b^2+c^2):\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\ge a^2+b^2+c^2$$

Therefore, it's minimum is $\frac{3}{5}$.

Solution 2 by Michael Sterghiou-Greece

$$P = \sum_{cyc} \frac{a^4}{b^5\sqrt{5(a^2+4)}} \text{ (1)}$$

Let $(p,q,r) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)$. $r = 1$. As $\sqrt{5(a^2+4)} \le \frac{1}{2}(a^2+9)$ (AM-GM)

and using BCS in (1) in the form $\sum_{cyc} \frac{\left(\frac{a^2}{b^2}\right)^2}{b\sqrt{s(a^2+4)}} = P$

we get $P \ge \frac{2\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2}{b\sum_{cyc}(a^2+9)}$ (2)

We will show RHS of (2) $\ge \frac{3}{5}$. We know that for $\begin{Bmatrix} xyz = 1 \\ x,y,z > 0 \end{Bmatrix} \sum_{cyc}\frac{x}{y} \ge x+y+z$ so,

$\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2 \ge \left(\sum_{cyc} a^2\right)^2 \ge \left[\frac{\left(\sum_{cyc} a\right)^2}{3}\right]^2 = \frac{p^4}{9}$ (2) reduces to the (stronger) inequality

$\frac{\frac{1}{9}p^4}{\left(\sum_{cyc} ac^2\right)+9p} \ge \frac{3}{10}$ or $\frac{10}{9}p^4 - 3\sum_{cyc} ac^3 - 27p \ge 0$. But $\sum_{cyc} ac^2 \le \sum_{cyc} a^3$

(rearrangement) so it suffices to show that

$\frac{10}{9}p^4 - 3(p^3 - 3pq + 3) - 27p \ge 0$ or $\frac{10}{9}p^4 - 3p^3 - 27p + 9(pq-1) \ge 0$

As $pq \ge 9$ (for $r = 1, p \ge 3, q \ge 3$) it is enough that $\frac{10}{9}p^4 - 3p^3 - 27p + 72 \ge 0$

or $10p^4 - 27p^3 - 243p + 648 \ge 0$ or $(p-3)(10p^3 + 3p^2 + 9p - 216) \ge 0$

which clearly holds for $p \ge 3$. Done! $\left[P_{\min} = \frac{3}{5}\right]$

SP.172. Prove that for any real numbers x, y, z:

$$(x+y+z)(y+z-x)(z+x-y)(x+y-z) \le 4y^2z^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

Lemma: For $x, y, z \ge 0$ we have:

$$(x+y-z)(y+z-x)(z+x-y) \le xyz \text{ (1)}$$

Let $P(x,y,z) = (x+y+z)(x+y-z)(y+z-x)(z+x-y) - 4y^2z^2$

$\Rightarrow P(\pm x, \pm y, \pm z) = P(x, y, z) \forall x, y, z \in \mathbb{R}$

$\Rightarrow$ ***Suppose*** $x, y, z \geq 0$ ***and*** $x \leq y \leq z$***. Then,***

$$(x + y + z)(x + y - z)(y + z - x)(z + x - y) \overset{(1)}{\leq} (x + y + z)xyz$$

Must show that: $(x + y + z)xyz \leq 4y^2z^2$

$\Leftrightarrow x(x + y + z) \leq 4yz$ ***(*)***

$$x(x + y + z) \leq y(2y + z) \overset{(2)}{\leq} 4yz$$

$\because$ ***If*** $y = 0$ ***then (2) true.*** $\because$ ***If*** $y > 0$ ***then (2)*** $\Leftrightarrow 2y + z \leq 4z$

$\Leftrightarrow 2y \leq 3z$ ***(true because*** $0 < y \leq z \Rightarrow 2y \leq 2z < 3z$***)***

Now, we must show (1) true:

If $x + y - z < 0$ ***or*** $x + z - y < 0$ ***or*** $y + z - x < 0$ ***then (1) true.***

If $\begin{cases} x + y - z < 0 \\ x + z - y < 0 \end{cases} \Rightarrow 2x < 0 \Rightarrow x < 0$ ***(contrary)***

(etc).

If $x + y - z, x + z - y, y + z - x > 0$ ***then***

$$(m + n)(n + p)(p + m) \overset{(Cauchy)}{\geq} 8mnp(m, n, p > 0)$$

Let $m = x + y - z, n = x + z - y, p = y + z - x$***, we proved.***

Solution 2 by Michael Sterghiou-Greece

$$(x + y + z)(y + z - x)(z + x - y)(x + y - z) \leq 4y^2z^2 \quad \textit{(1)}$$

(1) reduces to $-(x^2 + y^2 - z^2)^2 \leq 0$ ***which is true.***

SP.173. Prove that for any positive real numbers x, y, z:

$$\frac{x^2\sqrt{y^2 + z^2} + y^2\sqrt{z^2 + x^2} + z^2\sqrt{x^2 + y^2}}{x^3 + y^3 + z^3} \leq \sqrt{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Amit Dutta-Jamshedpur-India

$$\rightarrow x^2\sqrt{y^2 + z^2} = \sqrt{2}x^3\sqrt{\frac{y^2 + z^2}{2x^2}} \Rightarrow \sqrt{2}x^3\sqrt{\frac{y^2 + z^2}{2x^2}} \overset{GM \leq AM}{\leq} \sqrt{2}x^3\left(\frac{y^2 + z^2 + 2x^2}{4x^2}\right)$$

$$\leq \frac{1}{2\sqrt{2}}x(y^2+z^2+2x^2)$$

$$\Rightarrow x^2\sqrt{y^2+z^2}=\sqrt{2}x^3\sqrt{\frac{y^2+z^2}{2x^2}}\leq \frac{x}{2\sqrt{2}}(y^2+z^2+2x^2)$$

$$\Rightarrow \sum_{cyc(x,y,z)} x^2\sqrt{y^2+z^2}\leq \sum_{cyc(x,y,z)}\frac{x}{2\sqrt{2}}(y^2+z^2+2x^2)\leq \frac{1}{2\sqrt{2}}\sum(xy^2+xz^2+2x^3)$$

Now, $\because (x-y)^2\geq 0\Rightarrow x^2+y^2-xy\geq xy\Rightarrow (x+y)(x^2+y^2-xy)\geq xy(x+y)$

$$\Rightarrow (x^3+y^3)\geq (x^2y+xy^2)\Rightarrow \sum_{cyc}(x^2y+xy^2)\leq \sum_{cyc}(x^3+y^3)\leq 2\sum_{cyc}x^3$$

$$\Rightarrow \sum_{cyc(x,y,z)} x^2\sqrt{y^2+z^2}\leq \frac{1}{2\sqrt{2}}\left(2\sum x^3+2x^3\right)\leq \left(\frac{4\sum x^3}{2\sqrt{2}}\right)$$

$$\therefore \sum_{cyc(x,y,z)} x^2\sqrt{y^2+z^2}\leq \sqrt{2}(x^3+y^3+z^3)$$

$$\Rightarrow \frac{x^2\sqrt{y^2+z^2}+y^2\sqrt{z^2+x^2}+z^2\sqrt{x^2+y^2}}{(x^3+y^3+z^3)}\leq \sqrt{2}$$

(proved) Equality when $x=y=z$.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know, $2\sum_{cyc}x^3\geq \sum_{cyc}xy(x+y)$

$$\frac{\sum_{cyc}x^2\sqrt{y^2+z^2}}{\sum_{cyc}x^3}=\frac{\sum_{cyc}x^{\frac{3}{2}}\sqrt{xy^2+xz^2}}{\sum_{cyc}x^3}\overset{CAUCHY\ SCHWARZ}{\leq}\frac{\sqrt{\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}xy(x+y)\right)}}{\sum_{cyc}x^3}$$

$$\leq \frac{\sqrt{2\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x^3\right)}}{\sum_{cyc}x^3}=\sqrt{2}\ \textit{(proved)}$$

Solution 3 by Tran Hong-Vietnam

Let $x^2+y^2+z^2=3$.

$$\because x^2\sqrt{3-x^2}\leq \frac{3x+x^3}{2\sqrt{2}}\Leftrightarrow \frac{\left(x^2-1\right)^2}{2\sqrt{2}}\geq 0\ \textit{(true)}$$

$$\Rightarrow \sum x^2\sqrt{3-x^2}\leq \frac{3(x+y+z)+\left(x^3+y^3+z^3\right)}{2\sqrt{2}}\ (*)$$

$$3(x+y+z)=(x+y+z)(x^2+y^2+z^2)\overset{(Chebyshev)}{\leq} 3(x^3+y^3+z^3)$$

$\Leftrightarrow x + y + z \le x^3 + y^3 + z^3$. Hence,

$$(*) \le \frac{4(x^3+y^3+z^3)}{2\sqrt{2}} = \sqrt{2}(x^3 + y^3 + z^3). \text{ Proved.}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For all $a, b, c > 0$, we have this fact $\sqrt{\frac{2(a^6+a^3b^3+a^3c^3)}{3}} \ge \sqrt{a^4a^2 + a^4c^2}$

If $\frac{2(a^6+a^3b^3+a^3c^3)}{3} \ge a^4b^2 + a^4c^2$

If $2(a^4 + ab^3 + ac^3) \ge 3(a^2b^2 + a^2c^2)$ and it's true. Because

$a^4 + ab^3 + ab^3 \ge 3a^2b^2, a^4 + ac^3 + ac^3 \ge 3a^2c^2$. Consider for $x, y, z > 0$, we get

that $\frac{x^2\sqrt{y^2+z^2}+y^2\sqrt{z^2+x^2}+z^2\sqrt{x^2+y^2}}{x^3+y^3+z^3} \le \sqrt{2}$

If $\sqrt{x^4y^2 + x^4z^2} + \sqrt{y^4z^2 + y^4x^2} + \sqrt{z^6x^2 + z^4y^2} \le \sqrt{2}(x^3 + y^3 + z^3)$

$$= \sqrt{2(x^3 + y^3 + z^3)^2}$$

$$= \sqrt{2(x^6 + y^6 + z^6) + 2((xy)^3 + (yz)^3 + (zx)^3)}$$

$$= \sqrt{\frac{3x^2}{3}\left((x^6 + x^3y^3 + x^3z^3) + (y^3 + y^3x^3 + y^3z^3) + (z^3 + z^3x^3 + z^3y^3)\right)}$$

If $\sqrt{x^4y^2 + x^4z^2} + \sqrt{y^4x^2 + y^4z^2} + \sqrt{z^4x^2 + z^4y^2} \le$

$$\le \sqrt{\frac{2(x^6 + x^3y^3 + x^3z^3)}{3}} + \sqrt{\frac{2(y^6 + y^3z^3 + y^3x^3)}{3}} + \sqrt{\frac{2(z^6 + z^3x^3 + z^3y^3)}{3}}$$

and it's true. Therefore, it's true.

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\sum x^2\sqrt{y^2 + z^2} = \sum x\sqrt{x^2y^2 + x^2z^2} \overset{CBS}{\le} \sqrt{\sum x^2}\sqrt{2\sum x^2y^2} \Rightarrow \frac{\sum x^2\sqrt{y^2 + z^2}}{\sum x^3}$$

$$\le \frac{\sqrt{\sum x^2}\sqrt{2\sum x^2y^2}}{\sum x^3} \overset{?}{\le} \sqrt{2} \Leftrightarrow \left(\sum x^3\right)^2 \overset{?}{\ge} \left(\sum x^2\right)\left(\sum x^2y^2\right)$$

$$\Leftrightarrow \sum x^6 + 2\sum x^3y^3 \underset{(1)}{\overset{?}{\ge}} \sum x^4y^2 + \sum x^2y^4 + 3x^2y^2z^2$$

Now, $\sum x^6 + 3x^2y^2z^2 \underset{(i)}{\overset{Schur}{\ge}} \sum x^4y^2 + \sum x^2y^4$ & $2\sum x^3y^3 \underset{(ii)}{\overset{A-G}{\ge}} 6x^2y^2z^2$

(i)+(ii)$\Rightarrow \sum x^6 + 3x^2y^2z^2 + 2\sum x^3y^3 \geq \sum x^4y^2 + \sum x^2y^4 + 6x^2y^2z^2$

$$\Rightarrow \sum x^6 + 2\sum x^3y^3 \geq \sum x^4y^2 + \sum x^2y^4 + 3x^2y^2z^2$$

$\Rightarrow$ ***(1) is true (Hence proved)***

SP.174. Prove that for any positive real numbers a, b, c, x, y, z:

$$(a^3 + 3x^3)(b^3 + 3y^3)(c^3 + 3z^3) \geq (ayz + bzx + cxy + xyz)^3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Nguyen Tan Phat-Vietnam

$$(a^3 + 3x^3)(b^3 + 3y^3)(c^3 + 3z^3) =$$

$$= (a^3 + x^3 + x^3 + x^3)(y^3 + b^3 + y^3 + y^3)(z^3 + z^3 + c^3 + z^3)$$

Using Holder's inequality, we have:

$$(a^3 + x^3 + x^3 + x^3)(y^3 + b^3 + y^3 + y^3)(z^3 + z^3 + c^3 + z^3) \geq (ayz + bzx + cxy + xyz)^3$$

$$\Rightarrow (a^3 + 3x^3)(b^3 + 3y^3)(c^3 + 3z^3) \geq (ayz + bzx + cxy + xyz)^3$$

Solution 2 by Tran Hong-Vietnam

Using Cauchy's inequality:

$$\because \frac{a^3}{(a^3+3x^3)} + \frac{y^3}{(b^3+3y^3)} + \frac{z^3}{(c^3+3z^3)} \geq \frac{3ayz}{\sqrt[3]{(a^3+3x^3)(b^3+3y^3)(c^3+3z^3)}} \quad \textbf{(1)}$$

$$\because \frac{x^3}{(a^3+3x^3)} + \frac{b^3}{(b^3+3y^3)} + \frac{z^3}{(c^3+3z^3)} \geq \frac{3bxz}{\sqrt[3]{(a^3+3x^3)(b^3+3y^3)(c^3+3z^3)}} \quad \textbf{(2)}$$

$$\because \frac{x^3}{(a^3+3x^3)} + \frac{y^3}{(b^3+3y^3)} + \frac{c^3}{(c^3+3z^3)} \geq \frac{3xyc}{\sqrt[3]{(a^3+3x^3)(b^3+3y^3)(c^3+3z^3)}} \quad \textbf{(3)}$$

$$\because \frac{x^3}{(a^3+3x^3)} + \frac{y^3}{(b^3+3y^3)} + \frac{z^3}{(c^3+3z^3)} \geq \frac{3xyz}{\sqrt[3]{(a^3+3x^3)(b^3+3y^3)(c^3+3z^3)}} \quad \textbf{(4)}$$

From (1)+(2) +(3) +(4) we have:

$\Rightarrow \sqrt[3]{(a^3 + 3x^2)(b^3 + 3y^3)(c^3 + 3z^3)} \geq (xyz + xyc + bxz + ayz) \Rightarrow$ ***Proved.***

SP.175. Let x, y, z be positive real numbers such that:

$x^2 + y^2 + z^2 + 2xyz = 1$. Find the minimum value of:

$$P = \frac{x^3}{1 + 3y - 2yz} + \frac{y^3}{1 + 3z - 2zx} + \frac{z^3}{1 + 3x - 2xy}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

$$LHS = \sum \frac{x^4}{x + 3xy - 2xyz} \overset{(Schwarz)}{\geq}$$

$$\frac{(\sum x^2)^2}{\sum x + 3\sum xy - 6xyz} = \frac{(\sum x^2)^2}{\sum x + 3\sum xy + 3\sum(\sum x^2 - 1)}$$

$$\geq \frac{(\sum x^2)^2}{\sqrt{3\sum x^2} + 6\sum x^2 - 3} = \frac{t^4}{\sqrt{3}t + 6t^2 - 3}; \left(t = \sqrt{\sum x^2}\right)$$

$$\therefore (x^2 + y^2 + z^2)^3 \geq 27x^2y^2z^2 \Rightarrow t^6 \geq 27(xyz)^2$$

$$\Leftrightarrow t^3 \geq 3\sqrt{3}xyz \Leftrightarrow xyz \leq \frac{t^3}{3\sqrt{3}}$$

$$\because 1 = 2xyz + \sum x^2 \leq 2 \cdot \frac{t^3}{3\sqrt{3}} + t^2 \Leftrightarrow t \geq \frac{\sqrt{3}}{2} \approx 0.8660$$

Let $f(t) = \frac{t^4}{\sqrt{3}t + 6t^2 - 3}; t \in \left[\frac{\sqrt{3}}{2}; +\infty\right) \Rightarrow f'(t) = \frac{3t^3(4t^2 + \sqrt{3}t - 4)}{(6t^2 + \sqrt{3}t - 3)^2}$

$$f'(t) = 0 \Leftrightarrow 4t^2 + \sqrt{3}t - 4 = 0 \Leftrightarrow \left[\begin{array}{l} t = \frac{\sqrt{67} - \sqrt{3}}{8} \approx 0,8067 \notin \left[\frac{\sqrt{3}}{2}; +\infty\right) \\ t = \frac{-\sqrt{67} - \sqrt{3}}{8} \notin \left[\frac{\sqrt{3}}{2}; +\infty\right) \end{array}\right.$$

$$\Rightarrow f'(t) > 0\ \forall t \geq \frac{\sqrt{3}}{2} \Rightarrow f(t) \nearrow \left[\frac{\sqrt{3}}{2}; +\infty\right)$$

$$\Rightarrow f(t) \geq f\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16} \Rightarrow P_{\min} = \frac{3}{16} \Leftrightarrow x = y = z = \frac{1}{2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum x^2 = 1 - 2xyz \overset{A-G}{\geq} 3\sqrt[3]{x^2y^2z^2} \Rightarrow 1 - 2p^3 \geq 3p^2 \text{ (where } p = \sqrt[3]{xyz}\text{)}$$

$$\Rightarrow 2p^3 + 3p^2 - 1 \leq 0 \Rightarrow (2p - 1)(p + 1)^2 \leq 0 \Rightarrow p \leq \frac{1}{2} \Rightarrow \sqrt[3]{xyz} \leq \frac{1}{2} \Rightarrow xyz \leq \frac{1}{8}$$

$$\Rightarrow -2xyz \ge -\frac{1}{4} \Rightarrow 1-2xyz = \sum x^2 \ge \frac{3}{4} \Rightarrow \sqrt{3\sum x^2} \overset{(1)}{\ge} \frac{3}{2}$$

Now, $1+3x-2xy = \sum x^2 + 2xyz + 3x - 2xy$

$= (x-y)^2 + z^2 + 2xyz + 3x > 0$ ($\because x, y, z > 0$)

Similarly, $1+3y-2yz > 0$ **&** $1+3z-2xz > 0$

$$\therefore p = \frac{x^4}{x+3xy-2xyz} + \frac{y^4}{y+3yz-2xyz} + \frac{z^4}{z+3zx-2xyz}$$

$$\overset{Bergstrom}{\ge} \frac{(\sum x^2)^2}{\sum x + 3\sum xy - 6xyz} \ge \frac{(\sum x^2)^2}{\sqrt{3x^2} + 3\sum x^2 - 6xyz}$$

$$\left(\because \left(\sum x\right)^2 \le 3\sum x^2 \,\&\, \sum xy \le \sum x^2\right)$$

$$= \frac{(\sum x^2)^2}{\sqrt{3\sum x^2} + 3\sum x^2 + 3\sum x^2 - 3} \left(\because -2xyz = \sum x^2 - 1\right)$$

$$= \frac{\left(\frac{t^2}{3}\right)^2}{t+t^2+t^2-3} \quad \textit{(where } t = \sqrt{3\sum x^2}\textit{)}$$

$$\overset{?}{\ge} \frac{3}{16} \Leftrightarrow \frac{t^4}{9(2t^2+t-3)} \overset{?}{\ge} \frac{3}{16} \Leftrightarrow 16t^4 \overset{?}{\ge} 27(2t^2+t-3)$$

$$\left(\begin{array}{l} \because 2t^2+t-3 = (t-1)(2t+3) > 0 \\ as\ t = \sqrt{3\sum x^2} \ge \frac{3}{2} > 1 \ (from\ (1)) \end{array}\right)$$

$$\Leftrightarrow 16t^4 - 54t^2 - 27t + 81 \overset{?}{\ge} 0 \Leftrightarrow (2t-3)(8t^3+12t^2-9t-27) \overset{?}{\ge} 0$$

$\rightarrow$ ***true*** $\because t = \sqrt{3\sum x^2} \overset{by(1)}{\ge} \frac{3}{2} \Rightarrow (2t-3) \ge 0$ **&** $8t^3+12t^2-9t-27$

$= (8t^3-27) + 3t(4t-3) = (2t-3)(4t^2+6t+9) + 3t(2(2t-3)+3)$

> 0 ***as*** $t \ge \frac{3}{2}$

$\therefore p \ge \frac{3}{16} \Rightarrow P_{\min} = \frac{3}{16}$ ***& the minimum occurs when*** $x=y=z$ **&** $3x^2 + 2x^3 = 1$

$\Rightarrow$ ***when*** $(2x-1)(x+1)^2 = 0 \Rightarrow$ ***when*** $x=y=z=\frac{1}{2}$

Solution 3 by Michael Sterghiou-Greece

$\left(\sum_{cyc} x^2\right) + 2xyz = 1$ ***(T)***

$$P=\sum_{cyc}\frac{x^3}{1+3y-2yz}\ \textit{(1)}$$

Let $(p,q,r,m)=\left(\sum_{cyc}x,\sum_{cyc}xy,xyz,\sum_{cyc}x^2\right)$

From (T): $m+2r=1$. *But* $m\ge 3r^{\frac{2}{3}}\to 3r^{\frac{2}{3}}+2r-1\le 0\to$

$\to\left(\sqrt[3]{r}+1\right)^2\left(2\sqrt[3]{r}-1\right)\le 0$ *or* $r\le\frac{1}{8}$ *which means* $m\ge\frac{3}{4}$ *and* $m<1$ *as* $r>0$. *We will*

show that $P\ge\frac{3}{16}$. *(1)* $\to P=\sum_{cyc}\frac{x^4}{x+3xyz-2xyz}\overset{(BCS)}{\ge}\frac{m^2}{p+3q-6r}\ge\frac{3}{16}$ *(2)*

But $m=p^2-2q$, $r=\frac{1-m}{2}$ *so, (2)* $\to 32m^2-9m-9p^2-6p+18\ge 0$ *(3)*

But $\frac{p^2}{3}\le m$ *so, (3) becomes the stronger inequality*

$32m^2-9m-27m-6\sqrt{3m}+18\ge 0$ *or* $32m^2-36m-6\sqrt{3m}+18\ge 0$ *(4) with*

$\frac{3}{4}\le m<1$. *Let* $m=\frac{t^2}{3}$ *with* $\frac{3}{2}\le t<\sqrt{3}$ *(4)*$\to 16t^4-54t^2-27t+81\ge 0$

or $(2t-3)\left[\frac{1}{2}(2t-3)(8t^2+24t+27)+\frac{27}{2}\right]\ge 0$ *which is true for* $t\ge\frac{3}{2}$

Equality for $x=y=z=\frac{1}{2}$. *Done!*

SP.176. Prove that if $m\in[0,\infty)$, $x,y,z,t\in(0,\infty)$, then in any triangle ABC, with the usual notations holds:

$$\sum_{cyc}\frac{\left(xm_a^2+ym_b^2\right)^{m+1}}{(zb^2+twc_c^2)^m}\ge\frac{3^{m+\frac{3}{2}}(x+y)^{m+1}}{(4z+3t)^m}S$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Tran Hong-Vietnam

We have: $\sum w_a^2\le\sum s(s-a)=s^2$ *(1)*

Then, $LHS\overset{Radon}{\ge}\frac{\left(x\sum m_a^2+y\sum m_a^2\right)^{m+1}}{\left(z\sum a^2+t\sum w_a^2\right)^m}$

$$=\frac{\left[(x+y)\sum m_a^2\right]^{m+1}}{(z\sum a^2+t\sum w_a^2)^m}=\frac{\left[(x+y)\cdot\frac{3}{4}\sum a^2\right]^{m+1}}{(z\sum a^2+t\sum w_a^2)^m}\overset{(1)}{\ge}\frac{\left(\frac{3}{4}\right)^{m+1}\cdot(x+y)^{m+1}\cdot\left(\sum a^2\right)^{m+1}}{(z\sum a^2+ts^2)^m}$$

$$\geq \frac{\left(\frac{3}{4}\right)^{m+1}(x+y)^{m+1}\left(\sum a^2\right)^{m+1}}{\left(za^2+\frac{3}{4}t\sum a^2\right)^m}\ \left(\because 4s^2=\left(\sum a\right)^2\leq 3\sum a^2\right)$$

$$=\frac{\left(\frac{3}{4}\right)^{m+1}(x+y)^{m+1}\left(\sum a^2\right)^{m+1}}{\left(z+\frac{3}{4}\right)^m\left(\sum a^2\right)^m}=\frac{3^{m+1}(x+y)^{m+1}}{4\cdot(4z+3t)^m}\cdot\left(\sum a^2\right)\overset{(Weitzenbock)}{\geq}$$

$$\frac{3^{m+1}(x+y)^{m+1}\cdot 4\sqrt{3}S}{4(4z+3t)^m}=\frac{3^{m+\frac{3}{2}}(x+y)^{m+1}S}{(4z+3t)^m}$$

(proved)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS\overset{Radon}{\geq}\frac{(x+y)^{m+1}\left(\sum m_a^2\right)^{m+1}}{\left(z\sum a^2+t\sum w_a^2\right)^m}\overset{\sum w_a^2\leq s^2}{\geq}\frac{(x+y)^{m+1}\left(\frac{3}{4}\right)^{m+1}\left(\sum a^2\right)^{m+1}}{\left(z\sum a^2+ts^2\right)^m}$$

$$\overset{4s^2\leq 3\sum a^2}{\geq}\frac{(x+y)^{m+1}\cdot 3^{m+1}\left(\sum a^2\right)^{m+1}}{4^{m+1}\left(z\sum a^2+t\cdot\frac{3}{4}\sum a^2\right)^m}=\frac{4^m(x+y)^{m+1}\cdot 3^{m+1}\cdot\left(\sum a^2\right)^{m+1}}{4^{m+1}(4z+3t)^m\left(\sum a^2\right)^m}$$

$$=\frac{3^{m+1}(x+y)^{m+1}\left(\sum a^2\right)}{4(4z+3t)^m}\overset{Ionescu-Weitzenbock}{\geq}\frac{3^{m+1}(x+y)^{m+1}4\sqrt{3}S}{4(4z+3t)^m}=$$

$$=\frac{3^{m+\frac{3}{2}}(x+y)^{m+1}}{(4z+3t)^m}S\ \textit{(Proved)}$$

SP.177. Prove that if $m\in[0,\infty), x,y,z,t\in(0,\infty)$, then in any triangle ABC, with the usual notations holds:

$$\sum_{cyc}\frac{\left(xa^2+ym_b^2\right)^{m+1}}{\left(zh_c^2+th_a^2\right)^m}\geq\frac{(4x+3y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Tran Hong-Vietnam

$$LHS\overset{(Radon)}{\geq}\frac{\left(x\sum a^2+y\sum m_b^2\right)^{m+1}}{\left\{(z+t)\sum h_a^2\right\}^m}$$

$$\geq \frac{\left(x\sum a^2 + y\sum m_b^2\right)^{m+1}}{(z+t)^m\left(\sum w_a^2\right)^m} \left(\because h_a \leq m_a, etc \Rightarrow \sum h_a^2 \leq \sum m_a^2\right)$$

$$\geq \frac{\left(x\sum a^2 + \frac{3}{4}y\sum a^2\right)^{m+1}}{(a+t)^m(s^2)^m} \left(\because \sum w_a^2 \leq \sum s\,(s-a) = s^2\right)$$

$$= \frac{\left(\sum a^2\right)^{m+1}(4x+3y)^{m+1}}{4(z+t)^m(4s^2)^m} \geq \frac{\left(\sum a^2\right)^{m+1}(4x+3y)^{m+1}}{4(z+t)^m\left(3\sum a^2\right)^m}$$

$$= \frac{\left(\sum a^2\right)(4x+3y)^{m+1}}{4\cdot 3^m(z+t)^m} \overset{Finsler-Hadwiger}{\geq} \frac{4\sqrt{3}S(4x+3y)^{m+1}}{4\cdot 3^m(z+t)^m} = \frac{(4x+3y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}\cdot S$$

(proved)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS \overset{Radon}{\geq} \frac{\left(x\sum a^2 + y\sum m_a^2\right)^{m+1}}{\left(z\sum h_a^2 + \sum h_a^2\right)^m} == \frac{\left(x\sum a^2 + y\cdot\frac{3}{4}\sum a^2\right)^{m+1}}{(z+t)^m\left(\sum h_a^2\right)^m} = \frac{(4x+3y)^{m+1}\left(\sum a^2\right)^{m+1}}{4^{m+1}(z+t)^m\left(4r^2s^2\sum\frac{1}{a^2}\right)^m}$$

$$\overset{Goldstone}{\geq} \frac{(4x+3y)^{m+1}\left(\sum a^2\right)^{m+1}}{4^{m+1}(z+t)^m\left(\frac{4r^2s^2\cdot 4R^2s^2}{16R^2r^2s^2}\right)^m} = \frac{(4x+3y)^{m+1}\left(\sum a^2\right)^{m+1}}{(4s^2)^m\cdot 4(z+t)^m}$$

$$\geq \frac{(4x+3y)^{m+1}\left(\sum a^2\right)^{m+1}}{\left(3\sum a^2\right)^m\cdot 4(z+t)^m} \left(\because 4s^2 = \left(\sum a\right)^2 \leq 3\sum a^2\right)$$

$$= \frac{(4x+3y)^{m+1}\left(\sum a^2\right)}{3^m\cdot 4(z+t)^m} \overset{Ionescu-Weitzenbock}{\geq} \frac{(4x+3y)^{m+1}\cdot 4\sqrt{3}S}{3^m\cdot 4(z+t)^m} = \frac{(4x+3y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S$$

(proved)

SP.178. Show that:

If $m \in [0,\infty)$, $x,y,z,t \in (0,\infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{\left(xa^2 + ym_b^2\right)^{m+1}}{(zm_c^2 + tm_a^2)^m} \geq \frac{(4x+3y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by the authors

By $\sum_{cyclic} m_a^2 = \frac{3}{4}\sum_{cyclic} a^2$***, and J. Radon's inequality, we obtain:***

$$\sum_{cyclic} \frac{\left(xa^2 + ym_b^2\right)^{m+1}}{(zm_c^2 + tm_a^2)^m} \overset{RADON}{\geq} \frac{\left(\sum_{cyclic}\left(xa^2 + ym_b^2\right)\right)^{m+1}}{\left(\sum_{cyclic}(zm_c^2 + tm_a^2)\right)^m} =$$

$$= \frac{\left(x\sum_{cyclic} a^2 + y\sum_{cyclic} m_b^2\right)^{m+1}}{(z+t)^m\left(\sum_{cyclic} m_a^2\right)^m} = \frac{\left(x\sum_{cyclic} a^2 + \frac{3y}{4}\sum_{cyclic} a^2\right)^{m+1}}{\left(\frac{3}{4}\right)^m (z+t)^m\left(\sum_{cyclic} a^2\right)^m} =$$

$$= \frac{(4x+3y)^{m+1}\left(\sum_{cyclic} a^2\right)^{m+1}}{4^{m+1}\left(\frac{3}{4}\right)^m (z+t)^m\left(\sum_{cyclic} a^2\right)^m} = \frac{(4x+3y)^{m+1}\left(\sum_{cyclic} a^2\right)}{4\cdot 3^m\cdot(z+t)^m} \quad \textit{(1)}$$

By Ion Ionescu – Weitzenböck inequality, we have: $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ ***(2)***

From (1) and (2) we obtain:

$$\sum_{cyclic} \frac{\left(xa^2 + ym_b^2\right)^{m+1}}{(zm_c^2 + tm_a^2)^m} \geq \frac{(4x+3y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m} S$$

Q.E.D.

SP.179. If $x \in [0,1)$ then:

$$\cos x \leq x^3 + \tan^3 x + \sin^{-1} x + e^x$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution by Tran Hong-Vietnam

Let $f(x) = x^3 + \tan^3 x + \sin^{-1} x + e^x - \cos x; \forall x \in [0,1)$

$$\Rightarrow f'(x) = 3x^2 + \frac{1}{\sqrt{1-x^2}} + \sin x + 3\tan^2 x \cdot \sec^2 x > 0, \forall x \in [0,1) \Rightarrow$$

$\Rightarrow f(x) \nearrow$ ***on*** $[0,1) \Rightarrow f(x) \geq f(0) = 0 \Rightarrow$ ***Proved.***

SP.180. If $x, y, z \in \mathbb{R}^+ \wedge x^2 + y^2 + z^2 = 3^n, n \in \mathbb{N}$ then:

$$\sqrt[4]{x+y} + \sqrt[4]{x+z} + \sqrt[4]{y+z} \leq \sqrt[4]{54\sqrt{3^{n+1}}}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution 1 by Amit Dutta-Jamshedpur-India

Using Power mean inequality, with $f_m = \sqrt[m]{\frac{a_1^m+a_2^m+\cdots+a_k^m}{k}}$

if $m > n \Rightarrow f_m \ge f_n$

$$\Rightarrow \sqrt[m]{\frac{a_1^m+a_2^m+\cdots+a_k^m}{k}} \ge \sqrt[n]{\frac{a_1^n+a_2^m+\cdots+a_k^n}{k}} \quad \textbf{(1)}$$

Putting $a_1 = (x+y), a_2 = (y+z), a_3 = (x+z)$

$$m = 1, n = \frac{1}{4}$$

$$\Rightarrow \frac{1}{3}[(x+y)+(y+z)+(x+z)] \ge \left\{\frac{\sqrt[4]{x+y}+\sqrt[4]{y+z}+\sqrt[4]{x+z}}{3}\right\}^4$$

$$\Rightarrow \left\{\frac{2(x+y+z)}{3}\right\}^{\frac{1}{4}} \ge \frac{1}{3}\left[\sqrt[4]{x+y}+\sqrt[4]{y+z}+\sqrt[4]{x+z}\right]$$

$$\Rightarrow \sqrt[4]{x+y}+\sqrt[4]{y+z}+\sqrt[4]{x+z} \le 3\left\{\frac{2(x+y+z)}{3}\right\}^{\frac{1}{4}}$$

Know that, $(x+y+z)^2 \le 3(x^2+y^2+z^2) \le 3(3^n) \Rightarrow (x+y+z) \le \sqrt{3^{n+1}} \Rightarrow$

$$\Rightarrow \sqrt[4]{x+y}+\sqrt[4]{y+z}+\sqrt[4]{x+z} \le \sqrt[4]{54\sqrt{3^{n+1}}}$$

(Proved)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \Rightarrow a+b \le \sqrt{\frac{4(a^2+b^2)}{2}}$$

$$\sum \sqrt[4]{x+y} \le \sum \sqrt[4]{\sqrt{4\cdot\frac{x^2+y^2}{2}}} = \sum \sqrt[4]{2}\cdot\sqrt[8]{\frac{x^2+y^2}{2}} =$$

$$= \sqrt[4]{2}\cdot\sum\sqrt[8]{\frac{x^2+y^2}{2}} \overset{CBS}{\le} \sqrt[4]{2}\cdot\sqrt{3\cdot\sum\sqrt[4]{\frac{x^2+y^2}{2}}} =$$

$$= \sqrt[4]{18}\cdot\sqrt{\sum\sqrt[4]{\frac{x^2+y^2}{2}}} \overset{CBS}{\leq} \sqrt[4]{18}\cdot\sqrt{\sqrt{3\cdot\sum\sqrt{\frac{x^2+y^2}{2}}}} =$$

$$= \sqrt[4]{54}\cdot\sqrt[4]{\sum\sqrt{\frac{x^2+y^2}{2}}} \overset{CBS}{\leq} \sqrt[4]{54}\cdot\sqrt[8]{3\sum\frac{x^2+y^2}{2}} =$$

$$= \sqrt[4]{54}\cdot\sqrt[8]{\sum x^2\cdot 3} = \sqrt[4]{54\cdot\sqrt{3^{n+1}}}$$

$$x = y = z = \sqrt{3^{n-1}}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $n \in \mathbb{N}$ ***and*** $x^2+y^2+z^2 = 3^n \Rightarrow (x+y+z)^2 \leq 3(x^2+y^2+z^2) = 3^{n+1}$

$$\Rightarrow x+y+z \leq \sqrt{3^{n+1}} \Rightarrow \sqrt[4]{x+y}+\sqrt[4]{y+z}+\sqrt[4]{z+x}$$

$$= (x+y)^{\frac{1}{4}}+(y+z)^{\frac{1}{4}}+(z+x)^{\frac{1}{4}} \leq \frac{(x+y+y+z+z+x)^{\frac{1}{4}}}{3^{\frac{1}{4}-1}} = 3^{\frac{3}{4}}\left(2(x+y+z)\right)^{\frac{1}{4}}$$

$$\leq 3^{\frac{3}{4}} \times 2^{\frac{1}{4}} \times \left(3^{\frac{(n+1)}{2}}\right)^4 = (3^3 \times 2)^{\frac{1}{4}}\left(3^{\frac{x+1}{2}}\right)^4 = \sqrt[4]{54\sqrt{3^{n+1}}} \text{ ok}$$

Therefore, it is true.

Solution 4 by Soumava Chakraborty-Kolkata-India

$$LHS \overset{CBS}{\leq} \sqrt{3}\sqrt{\sum\sqrt{x+y}} \overset{CBS}{\leq} \sqrt{3}\sqrt{\sqrt{3}\sqrt{2\sum x}} \overset{CBS}{\leq} \sqrt{3}\sqrt{\sqrt{3}\sqrt{2}\sqrt{\sqrt{3}\sqrt{\sum x^2}}} =$$

$$= \sqrt{3}\sqrt{\sqrt{6}\sqrt{\sqrt{3}3^{\frac{n}{2}}}}\left(\because \sum x^2 = 3^n\right)$$

$$= 3^{\frac{1}{2}}\sqrt{6^{\frac{1}{2}}3^{\frac{1}{4}}3^{\frac{n}{4}}} = 3^{\frac{1}{2}}\sqrt{2^{\frac{1}{2}}\cdot 3^{\left(\frac{3}{4}+\frac{n}{4}\right)}} = 3^{\frac{1}{2}}\sqrt{2^{\frac{1}{2}}\cdot 3^{\frac{n+1}{4}}\cdot 3^{\frac{1}{2}}} = \sqrt[4]{9}\sqrt[4]{6}\sqrt{\sqrt[4]{3^{n+1}}} = \sqrt[4]{54\cdot\sqrt{3^{n+1}}}$$

(proved)

Solution 5 by Tran Hong-Vietnam

Let $f(t)=\sqrt[4]{t}, t>0 \Rightarrow f'(t)=-\frac{3}{16}t^{\frac{-7}{4}}<0 \forall t>0$

Using Jensen's inequality, we have

$$f(x+y)+f(x+z)+f(y+z)\le 3f\left(\frac{2(x+y+z)}{3}\right)=3\sqrt[4]{\frac{2}{3}(x+y+z)}$$

$$\overset{BCS}{\le} 3\sqrt[4]{\frac{2}{3}\sqrt{3(x^2+y^2+z^2)}}=3\sqrt[4]{\frac{2}{3}\sqrt{3^{n+1}}}=\sqrt[4]{54\sqrt{3^{n+1}}}$$

(proved)

Solution 6 by Marian Ursărescu-Romania

We must show: $\left(\sqrt[4]{x+y}+\sqrt[4]{x+z}+\sqrt[4]{y+z}\right)^4\le 54\sqrt{3^{n+1}}$ ***(1)***

From Hölder's inequality, we have:

$$\left(\sqrt[4]{x+y}\right)^4+\left(\sqrt[4]{x+z}\right)^4+\left(\sqrt[4]{y+z}\right)^4\ge\frac{\left(\sqrt[4]{x+y}+\sqrt[4]{x+z}+\sqrt[4]{y+z}\right)^4}{2z}\Leftrightarrow$$

$$\left(\sqrt[4]{x+y}+\sqrt[4]{x+z}+\sqrt[4]{y+z}\right)^4\le 54(x+y+z)\quad \textbf{(2)}$$

From (1)+(2) we must show: $x+y+z\le\sqrt{3^{n+1}}\Leftrightarrow(x+y+z)^2\le 3^{n+1}$ ***(3)***

From Cauchy's inequality, we have:

$3(x^2+y^2+z^2)\ge(x+y+z)^2\Rightarrow(x+y+z)^2\le 3^{n+1}\Rightarrow$ ***(3) it's true.***

UP.166. Solve the equation in $\mathbb{R}$:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Amit Dutta-Jamshedpur-India

$$\textit{Domain}\rightarrow\begin{cases}x^3-2x^2+2x>0\\ 4x-3x^4>0\end{cases}$$

$$x^3-2x^2+2x=x(x^2-2x+2)=x[(x-1)^2+1]$$

$$\because x^3-2x^2+2x>0\Rightarrow x[(x-1)^2+1]>0\Rightarrow x>0$$

$$GM\le AM\ \sqrt{x^2-2x^2+2x}\le\frac{(x^2-2x^2+2x)+1}{2}$$

$$\sqrt{x^3 - 2x^2 + 2x} \leq \left(\frac{x^2 - 2x^2 + 2x + 1}{2}\right) \quad \textbf{(a)}$$

Equality holds when $x^2 - 2x^2 + 2x = 1$ ***(1)***

Again, using GM ≤ AM

$$3\sqrt[3]{x^2 - x + 1} \leq (x^2 - x + 1) + 1 + 1 \leq (x^2 - x + 3) \quad \textbf{(2)}$$

Equality holds when $x^2 - x + 1 = 1$ ***(2)***

Again, using GM ≤ AM

$$2\sqrt[4]{4x - 3x^4} \leq 2\left\{\frac{(4x - 3x^4) + 1 + 1 + 1}{4}\right\}$$

$$\leq \left(\frac{4x - 3x^4 + 3}{2}\right) \quad \textbf{(3)}$$

Equality holds when $4x - 3x^4 = 1$ ***(3)***

Adding (1), (2), (3):

$$\sqrt{x^3 - 2x^2 + 2x} + 3\sqrt[3]{x^2 - x + 1} + 2\sqrt[4]{4x - 3x^4} \leq$$

$$\leq \left(\frac{x^3 - 2x^2 + 2x + 1}{2}\right) + (x^2 - x + 3) + \left(\frac{4x - 3x^4 + 3}{2}\right)$$

$$\Rightarrow \frac{x^4 - 3x^3}{2} + 7 \leq \frac{-3x^4 + 4x + 10 + x^3}{2}$$

$$\Rightarrow x^4 - 3x^3 + 14 \leq -3x^4 + 4x + 10 + x^3$$

$$\Rightarrow 4x^4 - 4x^3 - 4x + 4 \leq 0$$

$$\Rightarrow x^4 - x^3 - x + 1 \leq 0 \Rightarrow x^3(x - 1) - 1(x - 1) \leq 0$$

$$\Rightarrow (x^3 - 1)(x - 1) \leq 0 \Rightarrow (x - 1)(x^2 + x + 1)(x - 1) \leq 0$$

$$\Rightarrow (x - 1)^2(x^2 + x + 1) \leq 0$$

$$\because x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \Rightarrow (x - 1)^2 \leq 0$$

$$(x - 1)^2 = 0 \Rightarrow x = 1 \quad \textbf{(4)}$$

From (1), (2), (3) & (4):

The only real solution is $x = 1$***.***

UP.167. Let a, b, c be positive real numbers such that: $abc = 1$.

R M M

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Find the maximum value of:

$$P=\frac{1}{\sqrt[3]{3a^4-4a+2b^2+11}}+\frac{1}{\sqrt[3]{3b^4-4b+2c^2+11}}+\frac{1}{\sqrt[3]{3c^4-4c+2a^2+11}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

$$3a^4-4a+2b^2+11=\{a^4-4a^3+6a^2-4a+1\}+\{2a^4+4a^3-6a^2+10+2b^2\}$$

$$=(a-1)^4+2(a^4+2a^3-3a^2+b^2+5)$$

$$\geq 2(a^4+2a^3-4a^2+a^2+b^2+5)\overset{(*)}{\geq}4(a+ab+1)$$

$$(*)\Leftrightarrow a^4+2a^3-4a^2+2ab+5\geq 2(a+ab+1)$$

$$(*)\Leftrightarrow a^4+2a^3-4a^2+5\geq 2(a+1)$$

$$\Leftrightarrow a^4+2a^3-4a^2-2a+3\geq 0\Leftrightarrow (a-1)^2(a+1)(a+3)\geq 0\ \text{(true with } a>0)$$

Hence: $3a^4-4a+2b^2+11\geq 4(a+ab+1)$, *etc. Now,*

Let $f(t)=\sqrt[3]{t},t>0\Rightarrow f'(t)=-\frac{2}{9}t^{-\frac{5}{3}}<0\ (\forall t>0)$

Using Jensen's inequality, we have: $P\leq 3\sqrt[3]{\frac{Q}{3}}$

$$\therefore Q=\frac{1}{3a^4-4a+2b^2+11}+\frac{1}{3b^4-4b+2c^2+11}+\frac{1}{3c^4-4c+2a^2+11}$$

$$\leq\frac{1}{4}\left(\frac{1}{a+ab+1}+\frac{1}{b+bc+1}+\frac{1}{c+ca+1}\right)$$

$$=\frac{1}{4}\left(\frac{1}{a+ab+1}+\frac{a}{a+ab+1}+\frac{ab}{a+ab+1}\right)=\frac{1}{4}\left(\frac{1+a+ab}{1+a+ab}\right)=\frac{1}{4}$$

$$\Rightarrow P\leq 3\sqrt[3]{\frac{1}{4\cdot 3}}=\frac{3}{\sqrt[3]{12}}=\sqrt[3]{\frac{9}{4}}.\ \textit{Equality}\Leftrightarrow a=b=c=1.$$

Solution 2 by Michael Sterghiou-Greece

$$\textit{(1)}\ P=\sum_{cyc}\frac{1}{\sqrt[3]{3a^4-4a+2b^2+11}}$$

As $f(x)=\sqrt[3]{x},x>0$ *is concave* $\left(f''(x)=-\frac{2}{9}x^{\frac{5}{3}}\right)$ *we have by Jensen that*

$$P\leq 3\left[\frac{1}{3}\cdot\sum_{cyc}\frac{1}{3a^4-4a+2b^2+11}\right]^{\frac{1}{3}}\ \text{or}\ 3\left(\frac{P}{3}\right)^3\leq\sum_{cyc}\frac{1}{3a^4-4a+2b^2+11}.$$

Now we have successively

$$3a^4+3\ge 6a^2\to \frac{P^3}{9}\le \sum_{cyc}\frac{1}{6a^2-4a+2b^2+8}$$

$$2a^2+2b^2\ge 4ab\to \frac{P^3}{9}\le \sum_{cyc}\frac{1}{4a^2-4a+4ab+8}$$ ***or***

$\frac{4}{9}P^3\le \sum_{cyc}\frac{1}{a^2-a+ab+2}$. ***Also, $a^2+1\ge 2a$ so, the last inequality becomes***

$\frac{4}{9}P^3\le \sum_{cyc}\frac{1}{a+ab+1}=1$ ***because $abc=1$. Therefore $P\le \sqrt[3]{\frac{9}{4}}$ which is the required maximum for $a=b=c=1$.***

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

for $a,b,c>0$ and $abc=1$, we have

$$\frac{1}{\sqrt[3]{3a^4-4a+2b^2+11}}+\frac{1}{\sqrt[3]{3b^4-4b+2c^2+11}}+\frac{1}{\sqrt[3]{3c^4-4c+2a^2+11}}\le \sqrt[3]{\frac{9}{4}}$$

If $\sqrt{3^2\left(\frac{1}{3a^4-4a+2b^2+1}+\frac{1}{3b^4-4b+2c^2+11}+\frac{1}{3c^4-4c+2a^2+11}\right)}\le \sqrt[3]{\frac{9}{4}}$

If $3^2\left(\frac{1}{3a^4-4a+2b^2+11}+\frac{1}{3b^4-4b+2c^2+11}+\frac{1}{3c^4-4c+2a^2+11}\right)\le \frac{9}{4}$

If $\frac{1}{2a^4+2b^2+8}+\frac{1}{2b^4+2c^2+8}+\frac{1}{2c^4+2a^2+8}\le \frac{1}{4}$

If $\frac{1}{a^4+b^2+4}+\frac{1}{b^4+c^2+4}+\frac{1}{c^4+a^2+4}\le \frac{1}{2}$

If $\frac{1}{2a^2+b^2+3}+\frac{1}{2b^2+c^2+3}+\frac{1}{2c^2+a^2+4}\le \frac{1}{2}$

If $\frac{1}{ab+a+1}+\frac{1}{bc+b+1}+\frac{1}{ca+c+1}\le 1$

If $\frac{1}{\frac{x}{z}+\frac{x}{y}+1}+\frac{1}{\frac{y}{x}+\frac{y}{z}+1}+\frac{1}{\frac{z}{y}+\frac{z}{x}+1}\le 1,\ a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x}$

If $\frac{yz}{xy+xz+yz}+\frac{xz}{yz+xy+xz}+\frac{xx}{zx+yz+xy}=\frac{xy+yz+zx}{xy+yz+zx}=1$ ***ok***

Therefore, it's true (Its maximum is $\sqrt[3]{\frac{9}{4}}$)

UP. 168. Let be $a>0$ and $f:(-\infty,-a-1)\cup(-a,+\infty)\to\mathbb{R}$;

$f(x)=\frac{1}{x^2+(2a+1)x+a^2+a}$. **Find:**

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$$\lim_{n\to\infty}\sqrt[n^2]{\left|\lim_{p\to\infty}\sum_{p\to\infty}^{p} f^{(n)}(k)\right|}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Naren Bhandari-Bajura-Nepal

For $a>0$, defined

$$f(x)=\frac{1}{(x^2+2xa+a^2)+(x+a)}=\frac{1}{(x+a)^2+(x+a)}=$$

$$=\frac{1}{(x+a+1)(x+a)}=\frac{1}{x+a}-\frac{1}{x+a+1}$$

Thus

$$f^n(x)=\underbrace{\frac{d}{dx}\left(\cdots\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{1}{x+a}-\frac{1}{x+a+1}\right)\right)\right)\cdots\right)}_{n\ times}$$

$$=\frac{(-1)^n n!}{(x+a)^{n+1}}-\frac{(-1)^n n!}{(x+a+1)^{n+1}}=(-1)^n n!\left[\frac{1}{(x+a)^{n+1}}-\frac{1}{(x+a+1)^{n+1}}\right]$$

replacing x by k and thus

$$\left(\left|\lim_{p\to\infty}\sum_{k=1}^{p} f^n(k)\right|_{>0}\right)^{\frac{1}{n^2}}=\lim_{p\to\infty}\left(\sum_{k=1}^{p}\left[\frac{n!}{(k+a)^{n+1}}-\frac{n!}{(k+a+1)^{n+1}}\right]\right)^{\frac{1}{n^2}}$$

Since the partial sum of

$$\sum_{k=1}^{p}\left[\frac{n!}{(k+a)^{n+1}}-\frac{n!}{(k+a+1)^{n+1}}\right]=\frac{n!}{(a+1)^{n+1}}-\frac{n!}{(p+a+1)^{n+1}}$$

As $p\to\infty$ and hence the

$$\lim_{p\to\infty}\sum_{k=1}^{p}\left[\frac{n!}{(k+1)^{n+1}}-\frac{n!}{(k+a+1)^{n+1}}\right]=\frac{n!}{(a+1)^{n+1}}-0$$

Finally, we obtain that

$$\lim_{p\to\infty}\sum_{k=1}^{p}\left[\frac{n!}{(k+a)^{n+1}}-\frac{n!}{(k+a+1)^{n+1}}\right]=\frac{n!}{(a+1)^{n+1}}-0$$

Finally, we obtain that

$$L=\left(\lim_{n\to\infty}\frac{n!}{(a+1)^{n+1}}\right)^{\frac{1}{n^2}}=\left(\lim_{n\to\infty}\frac{\sim\sqrt{2\pi n}}{(a+1)^{n+1}}\left(\frac{n}{e}\right)^{n}\right)^{\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}\frac{(2\pi)^{\frac{1}{2n^2}}\cdot n^{\frac{1+2n}{n^2}}}{e^{\frac{n}{n^2}}(a+1)^{\frac{n+1}{n^2}}}=\lim_{n\to\infty}\exp\left(\frac{(1+2n)\log n}{n^2}\right)=e^0=1$$

Solution 2 by Ravi Prakash-New Delhi-India

$$x^2+(2a+1)x+a^2+a=(x+a)^2+x+a=(x+a+1)(x+a)$$

$$\therefore f(x)=\frac{1}{x+a}-\frac{1}{x+a+1}$$

$$f^n(x)=\frac{(-1)^n n!}{(x+a)^{n+1}}-\frac{(-1)^n n!}{(x+a+1)^{n+1}}$$

$$\Rightarrow\sum_{k=1}^{p}f^n(k)=\sum_{k=1}^{p}\left[\frac{(-1)^n n!}{(k+a)^{n+1}}-\frac{(-1)^n n!}{(k+a+1)^{n+1}}\right]=$$

$$=(-1)^n n!\left[\frac{1}{(1+a)^{n+1}}-\frac{1}{(1+a+p)^{n+1}}\right]$$

$$\lim_{p\to\infty}\sum_{k=1}^{p}f^n(k)=\frac{(-1)^n n!}{(1+a)^{n+1}}$$

$$\left|\lim_{p\to\infty}\sum_{k=1}^{p}f^n(k)\right|=\frac{n!}{(1+a)^{n+1}}\Rightarrow\sqrt[n^2]{\left|\lim_{p\to\infty}\sum_{k=1}^{p}f^n(k)\right|}=\frac{(n!)^{\frac{1}{n^2}}}{(1+a)^{\frac{(n+1)}{n^2}}}$$

For $n\ge 2,\ 2^{n-1}\le n!\le n^n$

$$(2^{n-1})^{\frac{1}{n^2}}\le(n!)^{\frac{1}{n^2}}\le(n^n)^{\frac{1}{n^2}}$$

Or $2^{\frac{1}{n}-\frac{1}{n^2}}\le(n!)^{\frac{1}{n^2}}\le n^{\frac{1}{n}}$

$$\lim_{n\to\infty}2^{\frac{1}{n}-\frac{1}{n^2}}=1,\ \lim_{n\to\infty}n^{\frac{1}{n}}=1\ \therefore\ \lim_{n\to\infty}(n!)^{\frac{1}{n^2}}=1$$

Also, $\lim_{n\to\infty}(1+a)^{\frac{(n+1)}{n^2}}=\lim_{n\to\infty}(1+a)^{\frac{1}{n}+\frac{1}{n^2}}=1$

Thus, $\lim_{n\to\infty} n^2\sqrt{\left|\lim_{p\to\infty}\sum_{k=1}^{p} f^n(k)\right|} = \frac{1}{1} = 1$

Solution 3 by Remus Florin Stanca-Romania

Let be $a > 0$ ***and*** $f: (-\infty; -a-1) \cup (-a; +\infty) \to \mathbb{R}$

$f(x) = \frac{1}{x^2+(2a+1)x+a^2+a}$. ***Find:***

$$\Omega = \lim_{n\to\infty} \sqrt[n^2]{\left|\lim_{p\to\infty}\sum_{k=1}^{p} f^{(n)}(k)\right|}$$

$$f(x) = \frac{1}{x^2 + 2\cdot x\cdot\frac{2a+1}{2} + \frac{4a^2+4a+1}{4} + a^2 + a - \frac{4a^2+4a+1}{4}}$$

$$= \frac{1}{\left(x+\frac{2a+1}{2}\right)^2 - \frac{1}{4}} = \frac{1}{(x+a)(x+a+1)} = \frac{x+a+1-(x+a)}{(x+a)(x+a+1)} = \frac{1}{x+a} - \frac{1}{x+a+1}$$

$$\succ f^{(n)}(x) = n!\cdot(-1)^n\cdot\frac{1}{(x+a)^{n+1}} + (-1)^{n+1}\cdot n!\,\frac{1}{(x+a+1)^{n+1}}$$

$$\Rightarrow \sum_{k=1}^{p} f^{(n)}(k) = n!\cdot(-1)^n\cdot\left(\frac{1}{(a+1)^{n+1}} - \frac{1}{(p+a+1)^{n+1}}\right) \Rightarrow \left|\lim_{p\to\infty}\sum_{k=1}^{p} f^{(n)}(k)\right| =$$

$$= \frac{n!}{(a+1)^{n+1}} \Rightarrow \lim_{n\to\infty}\left(\frac{n!}{(a+1)^{n+1}}\right)^{\frac{1}{n^2}} = \lim_{n\to\infty} e^{\frac{\ln\left(\frac{n!}{(a+1)^{n+1}}\right)}{n^2}} = \lim_{n\to\infty} e^{\frac{\ln\frac{n+1}{a+1}}{2n+1}} =$$

$$= \lim_{n\to\infty} e^{\frac{\ln\frac{n+2}{n+1}}{2}} = e^0 = 1 \succ \Omega = 1$$

UP.169. Let be the sequence $x_1 > 0$ and $x_1^p + x_2^p + \cdots + x_n^p = \frac{1}{\sqrt[p+1]{x_{n+1}}}$,

$\forall n \in \mathbb{N}, p \in \mathbb{N}^*$. Find:

$$\lim_{n\to\infty} n^{p+1}\cdot x_n^{p^2+p+1}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\frac{1}{\sqrt[p+1]{x_{n+1}}}=\sum_{k=1}^{n}x_k^p=\frac{1}{\sqrt[p+1]{x_n}}+x_n^p\Rightarrow\frac{1}{\sqrt[p+1]{x_{n+1}}}-\frac{1}{\sqrt[p+1]{x_n}}=x_n^p>0$$

$\therefore x_n > x_{n+1}$ ***for all*** $n\in\mathbb{N}$***, hence*** $\{x_n\}_{n=1}^{\infty}$ ***is decreasing, hence bounded***

Let $\lim_{n\to\infty}x_n=l$ ***then*** $\frac{1}{\sqrt[p+1]{l}}=\frac{1}{\sqrt[p+1]{l}}+l^p\Rightarrow l=0$

$$\Omega=\lim_{n\to\infty}n^{p+1}\cdot x_n^{p^2+p+1}\Rightarrow\sqrt[p+1]{\Omega}=\lim_{n\to\infty}\frac{n}{\dfrac{1}{x_n^{\frac{p^2+p+1}{p+1}}}}\overset{\substack{CAESARO\\STOLZ}}{=}\lim_{n\to\infty}\frac{n+1-n}{\dfrac{1}{x_n^{\frac{p^2+p+1}{p+1}}}-\dfrac{1}{x_n^{\frac{p^2+p+1}{p+1}}}}$$

$$=\lim_{n\to\infty}\frac{1}{\left(\dfrac{1}{\sqrt[p+1]{x_n}}+x_n^p\right)^{p^2+p+1}-\dfrac{1}{x_n^{\frac{p^2+p+1}{p+1}}}}=\lim_{n\to\infty}\frac{1}{\dfrac{\left(1+x_n^{\frac{p^2+p+1}{p+1}}\right)^{p^2+p+1}-1}{x_n^{\frac{p^2+p+1}{p+1}}}}$$

$$=\frac{1}{p^2+p+1}\Rightarrow\Omega=\frac{1}{(p^2+p+1)^{p+1}}\ \textit{(Answer)}$$

Solution 2 by Remus Florin Stanca-Romania

$$\lim_{n\to\infty}n^{p+1}x_n^{p^2+p+1}=\lim_{n\to\infty}n^{p+1}x_n^{(p+1)^2}\cdot\frac{1}{x_n^p}==\lim_{n\to\infty}\frac{n^{p+1}x_n^{(p+1)^2}x_n}{x_n^{p+1}}\quad\textit{(1)}$$

$$x_1^p+\dots+x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}\succ\frac{1}{\sqrt[p+1]{x_n}}+x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}$$

we prove by using the Mathematical induction that $x_n>0;\forall n\in\mathbb{N}$***:***

1. we prove that $P(0)$: ***"***$x_0>0$***" is true (true).***

2. we suppose that $P(n)$: ***"***$x_n>0$***" is true***

3. we prove that $P(n+1)$: ***"***$x_{n+1}>0$***" is true by using*** $P(n)$***:***

$\frac{1}{\sqrt[p+1]{x_{n+1}}}=x_n^p+\frac{1}{\sqrt[p+1]{x_n}};x_n>0\Rightarrow\frac{1}{\sqrt[p+1]{x_{n+1}}}>0\Rightarrow x_{n+1}>0\Rightarrow$ ***true*** $\Rightarrow x_n>0;\forall n\in\mathbb{N}$

$$\frac{1}{\sqrt[p+1]{x_n}} + x_n^p = \frac{1}{\sqrt[p+1]{x_{n+1}}} \succ \frac{1}{\sqrt[p+1]{x_{n+1}}} - \frac{1}{\sqrt[p+1]{x_n}} = x_n^p > 0 \succ \sqrt[p+1]{x_{n+1}} < \sqrt[p+1]{x_n}$$

$\succ x_{n+1} < x_n \succ (x_n)_{n\in\mathbb{N}}$ ***is a decreasing sequence,*** $x_n > 0 \succ \exists l \in \mathbb{R}$ ***such that:***

$$\lim_{n\to\infty} x_n = l = \frac{1}{\sqrt[p+1]{l}} = l^p + \frac{1}{\sqrt[p+1]{l}} \Rightarrow l = 0 \Rightarrow \lim_{n\to\infty} x_n = 0$$

$$\textbf{\textit{(1)}} \Rightarrow \lim_{n\to\infty} n^{p+1} x_n^{p^2+p+1} = \lim_{n\to\infty} \left(\frac{x_n^{p+1}\cdot n\cdot \sqrt[p+1]{x_n}}{x_n}\right)^{p+1}$$

$$= \lim_{n\to\infty}\left(x_n^p \cdot n \cdot \sqrt[p+1]{x_n}\right)^{p+1} = L^{p+1}$$

$$x_n^p = \frac{1}{\sqrt[p+1]{x_{n+1}}} - \frac{1}{\sqrt[p+1]{x_n}} \Rightarrow L = \lim_{n\to\infty}\left(\left(\frac{1}{\sqrt[p+1]{x_{n+1}}} - \frac{1}{\sqrt[p+1]{x_{n+1}}}\right)\cdot n \cdot \sqrt[p+1]{x_n}\right) =$$

$$\lim_{n\to\infty}\left(n\cdot\left(\sqrt[p+1]{\frac{x_n}{x_{n+1}}} - 1\right)\right) \quad \textbf{\textit{(2)}}$$

$$x_n^p = \frac{1}{\sqrt[p+1]{x_{n+1}}} - \frac{1}{\sqrt[p+1]{x_n}} \Rightarrow \lim_{n\to\infty}\sqrt[p+1]{\frac{x_{n+1}}{x_n}} = 1$$

$$\overset{(2)}{\Rightarrow} L = \lim_{n\to\infty} n\cdot\frac{\frac{x_n}{x_{n+1}} - 1}{\left(\sqrt[p+1]{\frac{x_n}{x_{n+1}}}\right)^0 + \cdots + \left(\sqrt[p+1]{\frac{x_n}{x_{n+1}}}\right)^p} = \frac{1}{p+1}\cdot\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) =$$

$$= \frac{1}{p+1}\cdot\lim_{n\to\infty}\left(\left(x_n^{\frac{p^2+p+1}{p+1}}\right)^{p+1} - 1\right)$$

$$= \frac{1}{p+1}\cdot\lim_{n\to\infty} n\cdot x_n^{\frac{p^2+p+1}{p+1}}(p+1) = \lim_{n\to\infty}\frac{n}{x_n^{-\frac{p^2+p+1}{p+1}}} \overset{Stolz\ Cesaro}{=}$$

$$= \lim_{n\to\infty}\frac{1}{x_{n+1}^{-\frac{p^2+p+1}{p+1}} - x_n^{-\frac{p^2+p+1}{p+1}}}$$

$$\lim_{n\to\infty} x_{n+1}^{-\frac{p^2+p+1}{p+1}} - x_n^{-\frac{p^2+p+1}{p+1}} = \lim_{n\to\infty}\left(x_n^p + \frac{1}{\sqrt[p+1]{x_n}}\right)^{p^2+p+1} - x_n^{-\frac{p^2+p+1}{p+1}} =$$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{x_n}}\right)^{p^2+p+1}\cdot\left(\left(\frac{x_n^p+\frac{1}{\sqrt[p+1]{x_n}}}{\frac{1}{\sqrt[p+1]{x_n}}}\right)^{p^2+p+1}-1\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{x_n}}\right)^{p^2+p+1}\cdot\frac{\left(x_n^{p+\frac{1}{p+1}}+1\right)^{p^2+p+1}-1}{x_n^{p+\frac{1}{p+1}}}\cdot x_n^{p+\frac{1}{p+1}} =$$

$$= (p^2+p+1)\cdot\lim_{n\to\infty} x_n^{\frac{p^2+p+1}{p+1}}\cdot x_n^{-\frac{p^2+p+1}{p+1}} = p^2+p+1$$

$$\Rightarrow L = \frac{1}{p^2+p+1} \Rightarrow \lim_{n\to\infty} n^{p+1}\cdot x_n^{p^2+p+1} = \frac{1}{(p^2+p+1)^{p+1}}$$

UP.170. Find:

$$\lim_{n\to\infty}\int_0^1 \frac{arctan(nx)\,ln(1+x)}{1+x^2}dx$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Sagar Kumar-Patna Bihar-India

$$\lim_{n\to\infty}\int_0^1 \frac{\tan^{-1}(nx)\ln(1+x)}{(1+x^2)}dx = I$$

$$I = \frac{\pi}{2}\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

Put $x = \tan\theta$

$$dx = \sec^2\theta\, d\theta$$

$$I = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) \quad \textbf{(1)}$$

$$I = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right) \quad \textbf{(2)}$$

(1) + ***(2)***

$$I=\frac{\pi^2}{16}\ln 2$$

Solution 2 by Avishek Mitra-India

$$\Omega=\lim_{n\to\infty}\int_0^1 \tan^{-1}(nx)\frac{\ln(1+x)}{(1+x^2)}dx=\frac{\pi}{2}\int_0^1\frac{\ln(1+x)}{(1+x^2)}dx$$

Let $I=\int_0^1\frac{\ln(1+x)dx}{(1+x^2)}=[\ln(1+x)\cdot\tan^{-1}x]_0^1-\int_0^1\frac{\tan^{-1}x dx}{(1+x)}$

[let $x=\tan z\Rightarrow dx=\sec^2 z\,dz$***]***

$$=\frac{\pi}{4}\ln 2-\int_0^{\frac{\pi}{4}}\frac{z\cdot\sec^2 z\,dz}{(1+\tan z)}=\frac{\pi}{4}\ln 2-\int_0^{\frac{\pi}{4}}\frac{zdz}{\cos z(\sin z+\cos z)}$$

Let $I_1=\int_0^{\frac{\pi}{4}}\frac{zdz}{\cos z(\sin z+\cos z)}=\int_0^{\frac{\pi}{4}}\frac{\left(\frac{\pi}{4}-z\right)}{\cos\left(\frac{\pi}{4}-z\right)\left[\sin\left(\frac{\pi}{4}-z\right)+\cos\left(\frac{\pi}{4}-z\right)\right]}dz$

$$=\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{dz}{\cos z\,(\sin z+\cos z)}-I_1\Rightarrow$$

$$\Rightarrow 2I_1=\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{\sec^2 z\,dz}{(1+\tan z)}=\frac{\pi}{4}[\ln(1+\tan z)]_0^{\frac{\pi}{4}}\ln 2\Rightarrow I_1=\frac{\pi}{8}\ln 2$$

Hence $I=\frac{\pi}{4}\ln 2-\frac{\pi}{8}\ln 2=\frac{\pi}{8}\ln 2$

Hence $\Omega=\frac{\pi}{2}\cdot\frac{\pi}{8}\ln 2=\frac{\pi^2}{16}\ln 2$ ***(answer)***

Solution 3 by Abdul Mukhtar-Nigeria

$$\lim_{n\to\infty}\int_0^1\frac{\arctan(nx)\ln(1+x)}{1+x^2}dx=\int_0^1\left(\lim_{n\to 0}\tan^{-1}(nx)\right)\times\frac{\ln(1+x)}{1+x^2}dx$$

we know $n=\infty\Rightarrow\tan^{-1}(\infty\cdot x)=\tan^{-1}(\infty)=\frac{\pi}{2}\Rightarrow\frac{\pi}{2}\int_0^1\frac{\ln(1+x)}{1+x^2}dx$

let $x=\tan\theta$

$$dx=\sec^2\theta\,d\theta$$

$$\frac{\pi}{2}\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\, d\theta$$

Set $\phi = \frac{\pi}{4} - \theta$

$$= \frac{\pi}{2}\int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\frac{\pi}{4}-\phi\right)\right) - d\phi$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-\phi\right)\right) d\phi = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \left(1+\frac{1-\tan\phi}{1+\tan\phi}\right) d\phi$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\phi}\right) d\phi = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln 2\, d\phi - \int_0^{\frac{\pi}{4}} \ln(1+\tan\phi) \Rightarrow$$

$$\Rightarrow 2I = \frac{\pi}{4}\ln 2 \Rightarrow I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\phi)\ d\phi = \frac{\pi}{2}\left(\frac{\pi}{8}\ln 2\right) \Rightarrow \frac{\pi^2}{16}\ln 2$$

Solution 4 by Shivam Sharma-New Delhi-India

$$\Rightarrow \int_0^1 \left(\lim_{n\to\infty} ax\tan(nx)\right)\frac{\ln(1+x)}{1+x^2}dx \Rightarrow \frac{\pi}{2}\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

Let $x = \tan\theta \Rightarrow \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) d\theta \Rightarrow \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta \Rightarrow$

$$\Rightarrow \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln\left(1+\left(\frac{1-\tan\theta}{1+\tan\theta}\right)\right) d\theta \Rightarrow \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln(2)\, d\theta - \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\, d\theta$$

$$\Rightarrow \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \ln(2) - \Omega$$

(OR)

$$\Omega = \frac{\pi}{4}\ln(2)\int_0^{\frac{\pi}{4}} d\theta$$

(OR)

$$\Omega = \frac{\pi^2}{16}\ln(2) \quad \textit{(Answer)}$$

UP.171. Find that in any acute-angled ΔABC the following inequality holds:

$$\min\left(\frac{\sin A}{\sin B+\sin C},\frac{\sin B}{\sin A+\sin C},\frac{\sin C}{\sin A+\sin B}\right)\le\frac{\cos A+\cos B+\cos C}{3}\le$$
$$\le\max\left(\frac{\sin A}{\sin B+\sin C},\frac{\sin B}{\sin A+\sin C},\frac{\sin C}{\sin A+\sin B}\right)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Tran Hong-Vietnam

$$\textit{Let } T=\left\{\frac{\sin A}{\sin B+\sin C};\frac{\sin B}{\sin A+\sin C};\frac{\sin C}{\sin A+\sin B}\right\}$$

Suppose: $A\le B\le C\Rightarrow\begin{cases}a\le b\le c\\ \sin A\le\sin B\le\sin C\end{cases}$; *(with A,B,C: acute angled)*

We will prove that:

$$\min T\overset{(*)}{\le}\frac{\cos A+\cos B+\cos C}{3}\overset{(**)}{\le}\max T$$

First:

$$\because \min T=\frac{\sin A}{\sin B+\sin C}=\frac{a}{b+c}\quad (1)$$

$$\because \frac{\cos A+\cos B+\cos C}{3}=\frac{a^2b+b^2a+a^2c+c^2a+b^2c+c^2b-(a^3+b^3+c^3)}{6abc}\quad (2)$$

From (1) and (2) we have:

$$6bca^2\le(b+c)\{a^2b+b^2a+a^2c+c^2a+b^2c+c^2b-(a^3+b^3+c^3)\}$$
$$\Leftrightarrow (a+b+c)\{b(a^2-c^2)+c(a^2-b^2)-2a(b^2-bc+c^2)+b^3+c^3\}\le 0$$
$$\Leftrightarrow (a+b+c)\{b(a^2-c^2)+c(a^2-b^2)+(b^2-bc+c^2)(b+c-2a)\}\le 0$$
$$\Leftrightarrow (b-a)\{(b-c)^2-ac\}+(c-a)\{(b-c)^2-ab\}\le 0\quad (3)$$
$$(b-c)^2-ac\le(b-c)^2-a^2=-(b+a-c)(c+a-b)<0$$
$$\Rightarrow (b-a)\{(b-c)^2-ac\}\le 0\quad (4)$$

$$(b-c)^2-ab\le(b-c)^2-a^2=-(b+c-c)(c+a-b)<0$$

$$\Rightarrow (c-a)\{(b-c)^2-ab\}\le 0 \quad \textit{(5)}$$

$$\overset{(4),(5)}{\Rightarrow} \textit{ (3) true} \Rightarrow \textit{(*) true.}$$

$$\textit{Second: } \frac{\cos A+\cos B+\cos C}{3}\le\frac{\frac{3}{2}}{3}=\frac{1}{2}\quad \textit{(6)}$$

$$\max T=\frac{\sin C}{\sin A+\sin B}=\frac{c}{a+b}\ge\frac{1}{2}\quad \textit{(7)}$$

$$\overset{(6),(7)}{\Rightarrow} \textit{(**) true. Hence: For any acute-angled } \Delta ABC.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\min\left(\frac{\sin A}{\sin B+\sin C},\frac{\sin B}{\sin A+\sin C},\frac{\sin C}{\sin A+\sin B}\right)\overset{(1)}{\le}\frac{\sum\cos A}{3}\le$$

$$\overset{(2)}{\le}\max\left(\frac{\sin A}{\sin B+\sin C},\frac{\sin B}{\sin A+\sin C},\frac{\sin C}{\sin A+\sin B}\right)$$

$$\textit{RHS of (2)}\ge\frac{1}{3}\sum\frac{\sin A}{\sin B+\sin C}=\frac{1}{3}\sum\frac{a}{b+c}=$$

$$=\frac{1}{3}\frac{\sum a(c+a)(a+b)}{2s(s^2+2Rr+r^2)}=\frac{\sum a\left(\sum ab+a^2\right)}{3\cdot 2s(s^2+2Rr+r^2)}$$

$$=\frac{\left(\sum ab\right)(2s)+2s(s^2-6Rr-3r^2)}{3\cdot 2s(s^2+2Rr+r^2)}$$

$$=\frac{2s^2-2Rr-2r^2}{3(s^2+2Rr+r^2)}\overset{?}{\ge}\frac{\sum\cos A}{3}=\frac{R+r}{3R}$$

$$\Leftrightarrow R(2s^2-2Rr-2r^2)\overset{?}{\ge}(R+r)(s^2+2Rr+r^2)$$

$$\Leftrightarrow (R-r)s^2\overset{?}{\ge}(R+r)(2Rr+r^2)+R(2Rr+2r^2)$$

$$\overset{(2a)}{=}2R^2r+Rr^2+2Rr^2+r^3+2R^2r+2Rr^2=4R^2r+5Rr^2+r^3$$

$$\textit{Now, LHS of (2a)}\overset{Gerretsen}{\ge}(R-r)(16Rr-5r^2)\overset{?}{\ge}4R^2r+5Rr^2+r^3$$

$$\Leftrightarrow 16R^2-21Rr+5r^2\overset{?}{\ge}4R^2+5Rr+r^2$$

$$\Leftrightarrow 12R^2-26Rr+4r^2\overset{?}{\ge}0\Leftrightarrow 6R^2-13Rr+2r^2\overset{?}{\ge}0$$

$$\Leftrightarrow (R-2r)(6R-r)\overset{?}{\ge}0\to \textit{true} \because R\overset{Euler}{\ge}2r\Rightarrow \textit{(2a) \& hence (1) is true}$$

We shall now focus on proving (1), which is:

$$3\min\left(\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\right)\overset{(1a)}{\leq} 1+\frac{\Delta}{s}\cdot\frac{4\Delta}{abc}$$

$$=1+\frac{4s(s-a)(s-b)(s-c)}{sabc}=1+\frac{4xyz}{(x+y)(y+z)(z+x)}$$

(where $s-a=x, s-b=y, s-c=z$*)*

$$=\frac{6xyz+\sum x^2y+\sum xy^2}{(x+y)(y+z)(z+x)}$$

Case 1) $\min\left(\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\right)=\frac{a}{b+c}$

$$\therefore \frac{a}{b+c}\leq\frac{b}{c+a}\Rightarrow a\leq b\Rightarrow y+z\leq z+x\Rightarrow x\geq y$$

& $\frac{a}{b+c}\leq\frac{c}{a+b}\Rightarrow a\leq c\Rightarrow y+z\leq x+y\Rightarrow x\geq z$

Now, (1a) $\Leftrightarrow \frac{3(y+z)}{2x+y+z}\leq\frac{6xyz+\sum x^2y+\sum xy^2}{(x+y)(y+z)(z+x)}$

$$\Leftrightarrow x^3y+x^3z+4x^2yz-2y^2z^2-xy^3-xz^3-y^3z-yz^3\overset{(1b)}{\geq}0$$

Now, $2x^2yz-2y^2z^2=2yz(x^2-yz)\geq 0\ (\because x\geq yz),$

$$x^2yz-y^3z=yz(x^2-y^2)\geq 0\ (\because x\geq y),$$

$$x^2yz-yz^3=yz(x^2-z^2)\geq 0\ (\because x\geq z),$$

$$x^3y-xy^3=xy(x^2-y^2)\geq 0\ (\because x\geq y),$$

$$x^3z-xz^3=xz(x^2-z^2)\geq 0\ (\because x\geq z)$$

Adding the last 5 inequalities, (1b) & hence (1a) & hence (1) is true.

Case 2) $\min\left(\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\right)=\frac{b}{c+a}$

$\therefore \frac{b}{c+a}\leq\frac{a}{b+c}\Rightarrow b\leq a\Rightarrow z+x\leq y+z\Rightarrow y\geq x$ **&**

$$\frac{b}{c+a}\leq\frac{c}{a+b}\Rightarrow b\leq c\Rightarrow z+x\leq x+y\Rightarrow y\geq z$$

Now, (1a) $\Leftrightarrow \frac{3(z+x)}{x+2y+z}\leq\frac{6xyz+\sum x^2y+\sum xy^2}{(x+y)(y+z)(z+x)}$

$$\Leftrightarrow xy^3+y^3z+4xy^2z-2z^2x^2-xz^3-yz^3-x^3y-x^3z\overset{(1c)}{\geq}0$$

Now, $2xy^2z-2z^2x^2=2zx(y^2-zx)\geq 0\ (\because y\geq z,x),$

$$xy^2z-xz^3=zx(y^2-z^2)\geq 0\ (\because y\geq z),$$

$$xy^2z-x^3z=xz(y^2-x^2)\geq 0\ (\because y\geq x),$$

$$y^3z - yz^3 = yz(y^2 - z^2) \geq 0 \ (\because y \geq z),$$

$$xy^3 - x^3y = xy(y^2 - x^2) \geq 0 \ (\because y \geq x)$$

Adding the last 5 inequalities, (1c) & hence, (1a) & hence, (1) is true.

Case 3) $\min\left(\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}\right) = \frac{c}{a+b}$

$$\therefore \frac{c}{a+b} \leq \frac{a}{b+c} \Rightarrow c \leq a \Rightarrow x + y \leq y + z \Rightarrow z \geq x \ \&$$

$$\frac{c}{a+b} \leq \frac{b}{c+a} \Rightarrow c \leq b \Rightarrow x + y \leq z + x \Rightarrow z \geq y$$

Now, (1a) $\Leftrightarrow \frac{3(x+y)}{x+y+2z} \leq \frac{6xyz+\sum x^2y+\sum xy^2}{(x+y)(y+z)(z+x)}$

$$\Leftrightarrow yz^3 + zx^3 + 4xyz^2 - x^3y - x^3z - 2x^2y^2 - xy^3 - y^3z \overset{(1d)}{\geq} 0$$

Now, $2xyz^2 - 2x^2y^2 = 2xy(z^2 - xy) \geq 0 \ (\because z \geq x, y),$

$$xyz^2 - xy^3 = xy(z^2 - y^2) \geq 0 \ (\because z \geq x),$$

$$xyz^2 - x^3y = xy(z^2 - x^2) \geq 0 \ (\because z \geq x),$$

$$yz^3 - y^3z = yz(z^2 - y^2) \geq 0 \ (\because z \geq y)$$

$$xz^3 - x^3z = xz(z^2 - x^2) \geq 0 \ (\because z \geq x)$$

Adding the last 5 inequalities, (1d) & hence (1a) & hence (1) is true.

Combining the 3 cases, (1) is always true.

(This completes the proof)

UP.172. Let be $A \in M_5(\mathbb{R})$, invertible such that: $\det(A^2 + I_5) = 0$.

Prove that:

$$Tr\ A^* = 1 + \det A \cdot Tr\ A^{-1}$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

As $\det(A^2 + I_5) = 0$

$\det[(A + iI_5)(A - iI_5)] = 0 \Rightarrow \det(A + iI_5) = 0$ ***or*** $\det(A - iI_5) = 0$

$\Rightarrow i$ ***or*** $-i$ ***is an eigenvalue of*** A.

As characteristic equation of A it has real coefficients, both $i, -i$ are eigenvalues of A. Let $\lambda_1, \lambda_2, \lambda_3$ be other eigenvalues of A.

$$Tr(A^*) = (\lambda_1 + \lambda_2 + \lambda_3)(i - i) + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + i(-i)$$

$= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 1$. ***Also,*** $\det A = \lambda_1\lambda_2\lambda_3(i)(-i) = \lambda_1\lambda_2\lambda_3$

$$Tr(A^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{i} - \frac{1}{i} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}$$

$$\det(A)\, tr(A^{-1}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

Thus, $tr(A^*) = 1 + \det(A)\, tr(A^{-1})$

UP.173. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{6 - 2\sum_{i=2}^{n} \frac{1}{i+1}\binom{2i}{i} + 3\sum_{i=2}^{n}\binom{2i}{i}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$6 - 2\sum_{i=2}^{n} \frac{1}{i+1}\binom{2i}{i} + 3\sum_{i=2}^{n}\binom{2i}{i} = 6 + \sum_{i=2}^{n}\binom{2i}{i}\cdot\frac{3i+1}{i+1} =$$

$$= 6 + \sum_{i=2}^{n-1}\binom{2i}{i}\cdot\frac{3i+1}{i+1} + \binom{2n}{n}\cdot\frac{3n+1}{n+1} = 6 + \sum_{i=2}^{n-1}\binom{2i}{i}\cdot\frac{3i+1}{i+1} + \frac{(2n)!}{(n!)^2}\cdot\frac{3n+1}{n+1} \quad (1)$$

$$\lim_{n\to\infty}\frac{(2n)!}{(n!)^2}\cdot\frac{3n+1}{n+1} = 3\cdot\lim_{n\to\infty}\frac{(2n)!}{(n!)^2}$$

Let $x_n = \frac{(2n)!}{(n!)^2} \succ \frac{x_{n+1}}{x_n} = \frac{(2n+2)!}{\left((n+1)!\right)^2}\cdot\frac{(n!)^2}{(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \succ$

$$\succ \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = 4 > 1 \succ \lim_{n\to\infty} x_n = \infty \succ \lim_{n\to\infty}\frac{(2n)!}{(n!)^2} = \infty \succ$$

$$\Rightarrow \lim_{n\to\infty}\frac{(2n)!}{(n!)^2}\cdot\frac{3n+1}{n+1} = \infty \overset{(1)}{\Rightarrow} \lim_{n\to\infty}\left(6 - \sum_{i=2}^{n}\frac{1}{i+1}\cdot\binom{2i}{i} + 3\sum_{i=2}^{n}\binom{2i}{i}\right) = \infty \Rightarrow$$

$$\succ \lim_{n\to\infty}\sqrt[n]{6-2\sum_{i=2}^{n}\frac{1}{i+1}\cdot\binom{2i}{i}+3\sum_{i=2}^{n}\binom{2i}{i}}=$$

$$=\lim_{n\to\infty}\left(6-2\sum_{i=2}^{n}\frac{1}{i+1}\binom{2i}{i}+3\sum_{i=2}^{n}\binom{2i}{i}\right)^{\frac{1}{n}}\overset{\infty^0}{=}\lim_{n\to\infty}e^{\frac{\ln\left(6+\sum_{i=2}^{n}\binom{2i}{i}\cdot\frac{3i+1}{i+1}\right)}{n}}\overset{Stolz\ Cesaro}{=}$$

$$=\lim_{n\to\infty}e^{\ln\left(\frac{6+\sum_{i=2}^{n+1}\binom{2i}{i}\cdot\frac{3i+1}{i+1}}{6+\sum_{i=2}^{n}\binom{2i}{i}\cdot\frac{3i+1}{i+1}}\right)}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{\binom{2n+4}{n+2}\cdot\frac{3n+7}{n+3}}{\binom{2n+2}{n+1}\cdot\frac{3n+4}{n+2}}=$$

$$=\lim_{n\to\infty}\frac{(2n+4)!}{(n+2)!\,(n+2)!}\cdot\frac{3n+7}{n+3}\cdot\frac{n+2}{3n+4}\cdot\frac{(n+1)!\,(n+1)!}{(2n+2)!}=$$

$$=\lim_{n\to\infty}\frac{(2n+3)(2n+4)}{(n+2)(n+2)}=4\Rightarrow\Omega=4$$

Solution 2 by Pierre Mounir-Cairo-Egypt

$$L=\sqrt[n]{6-2\sum_{k=2}^{n}\left(\frac{1}{k+1}\right)\binom{2k}{k}+3\sum_{k=2}^{n}\binom{2k}{k}}$$

$$=\sqrt[n]{6+\sum_{k=2}^{n}\left(\frac{3k+1}{k+1}\right)\binom{2k}{k}}=\sqrt[n]{1+\sum_{k=0}^{n}\left(\frac{3k+1}{k+1}\right)\binom{2k}{k}}$$

Let $S_n=1+\sum_{k=0}^{n}\left(\frac{3k+1}{k+1}\right)\binom{2k}{k}\Rightarrow L=\lim_{n\to\infty}\sqrt[n]{S_n}$

$$\lim_{n\to\infty}\frac{S_{n+1}}{S_n}\overset{SC}{=}\lim_{n\to\infty}\frac{S_{n+1}-S_n}{S_n-S_{n-1}}\quad (S_n\to\infty\ \ as\ n\to\infty)$$

$$=\lim_{n\to\infty}\frac{\left(\frac{3n+4}{n+2}\right)\binom{2n+2}{n+1}}{\left(\frac{3n+1}{n+1}\right)\binom{2n}{n}}=\lim_{n\to\infty}\frac{(3n+4)(n+1)(2n+2)!\,(n!)^2}{(3n+1)(n+2)(2n)!\,(n+1)!^2}$$

$$=\lim_{n\to\infty}\frac{(3n+4)(n+1)(2n+2)(2n+1)}{(3n+1)(n+2)(n+1)^2}=4$$

$$\therefore L=\lim_{n\to\infty}\sqrt[n]{S_n}=\lim_{n\to\infty}\frac{S_{n+1}}{S_n}=4$$

UP.174. If $f:[a,b]\to[1,\infty);\ 0<a\le b;\ f$ integrable then:

$$\int_a^b\int_a^b\int_a^b \frac{3+f(x)+f(y)+f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(y)}\,dx\,dy\,dz \le (b-a)^3+\left(\int_a^b \frac{dx}{f(x)}\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$f(x);f(y);f(z)\in[1,\infty)\Rightarrow f(x)\ge 1;f(y)\ge 1$$

$$\Rightarrow f(x)(f(y)-1)+f(y)(f(x)-1)\ge 0$$

$$2f(x)f(y)-f(x)-f(y)\ge 0\quad (1)$$

$$f(x)\ge 1;f(y)\ge 1\Rightarrow f(x)f(y)-1\ge 0\quad (2)$$

By (1); (2):

$$\sum_{cyc}(f(x)f(y)-1)\big(2f(x)f(y)-f(x)-f(y)\big)\ge 0$$

$$\sum_{cyc}\left(\frac{f(x)+f(y)+2f^2(x)f^2(y)-2f(x)f(y)-f^2(x)f(y)-f(x)f^2(y)}{2f(x)f(y)}\right)\ge 0$$

$$\sum_{cyc}\left(\frac{f(x)+f(y)}{2f(x)f(y)}+f(x)f(y)-1-\frac{f(x)}{2}-\frac{f(y)}{2}\right)\ge 0$$

$$\sum_{cyc}\frac{f(x)+f(y)}{2f(x)f(y)}+\sum_{cyc}f(x)f(y)\ge 3+\sum_{cyc}f(x)$$

$$\sum_{cyc}\frac{1}{f(x)}+\sum_{cyc}f(x)f(y)\ge 3+\sum_{cyc}f(x)$$

$$\left(\sum_{cyc}f(x)f(y)\right)\frac{1}{f(x)f(y)f(z)}+\sum_{cyc}f(x)f(y)\ge 3+\sum_{cyc}f(x)$$

$$1+\frac{1}{f(x)f(y)f(z)}\ge\frac{3+f(x)+f(y)+f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)}$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{3+f(x)+f(y)+f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)}\right)dx\,dy\,dz\le$$

$$\leq \int_a^b\int_a^b\int_a^b dx\,dy\,dz + \int_a^b\int_a^b\int_a^b \left(\frac{dx\,dy\,dz}{f(x)f(y)f(z)}\right) == (b-a)^3 + \left(\int_a^b \frac{dx}{f(x)}\right)^3$$

UP.175. In acute ΔABC the following relationship holds:

$$\frac{b^2+c^2+2bc}{b^2+c^2-a^2} + \frac{c^2+a^2+2ca}{c^2+a^2-b^2} + \frac{a^2+b^2+2ab}{a^2+b^2-c^2} > 9$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

$$LHS = \frac{(b+c)^2}{b^2+c^2-a^2} + \frac{(c+a)^2}{c^2+a^2-b^2} + \frac{(a+b)^2}{a^2+b^2-c^2}$$

$$\geq \frac{4bc}{b^2+c^2-a^2} + \frac{4ca}{c^2+a^2-b^2} + \frac{4ab}{a^2+b^2-c^2}$$

$$= \frac{2}{\cos A} + \frac{2}{\cos B} + \frac{2}{\cos C} = 2\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right)$$

$$= 2 \cdot \frac{p^2+r^2-4R^2}{p^2-(2R-r)^2} \quad (*)$$

We prove

(*)≥ 12

$$\Leftrightarrow p^2+r^2-4R^2 \geq 6p^2-6(2R-r)^2 \Leftrightarrow p^2+r^2-4R^2 \geq 6p^2-6(4R^2-4Rr+r^2)$$

$$\Leftrightarrow 20R^2+7r^2+24Rr \geq 5p^2$$

Which is true because

$$p^2 \leq 4R^2+4Rr+3r^2 \Leftrightarrow 5p^2 \leq 20R^2+20Rr+15r^2 \overset{(1)}{\leq} 20R^2+24Rr+7r^2$$

(1) $\Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r \Rightarrow$ ***(*)*** $\geq 12 > 9$. ***Proved.***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{b^2+c^2+2bc}{b^2+c^2-a^2} + \frac{c^2+a^2+2ca}{c^2+a^2-b^2} + \frac{a^2+b^2+2ab}{a^2+b^2-c^2} \geq 12$$

Let $b^2+c^2-a^2 = x, c^2+a^2-b^2 = y, a^2+b^2-c^2 = z \therefore a^2 = \frac{y+z}{2}, b^2 = \frac{z+x}{2}, c^2 = \frac{x+y}{2}$

Now, LHS $\frac{b^2+c^2-a^2+a^2+2bc}{b^2+c^2-a^2} + \frac{c^2+a^2-b^2+b^2+2ca}{c^2+a^2-b^2} + \frac{a^2+b^2-c^2+c^2+2ab}{a^2+b^2-c^2} =$

$$= 3 + \sum \frac{a^2}{b^2 + c^2 - a^2} + \sum \frac{2bc}{b^2 + c^2 - a^2} = 3 + \sum \frac{a^2}{b^2 + c^2 - a^2} + \sum \frac{2bc}{2bc \cos A} =$$

$$= 3 + \sum \left(\frac{\frac{y+z}{2}}{x} \right) + \sum \frac{1}{\cos A}$$

(using above substitution) $\overset{Bergstrom}{\geq} 3 + \left(\frac{1}{2}\right) \sum \left(\frac{x}{y} + \frac{y}{x}\right) + \frac{9}{\sum \cos A} \overset{A-G}{\geq} 3 + 3 + \frac{9}{\sum \cos A} \geq 6 + \frac{9}{\frac{3}{2}}$

$\left(\because \sum \cos A \leq \frac{3}{2}\right) = 12 > 9$ ***(equality when ΔABC is equilateral)***

Solution 3 by Lahiru Samarakoon-Sri Lanka

$$\sum \frac{a^2 + b^2 + 2ba}{a^2 + b^2 - c^2} > 9$$

for acute ABC, $\cos A, \cos B, \cos C > 0$ then:

$$LHS = \sum \frac{a^2 + b^2 + 2ba}{a^2 + b^2 - c^2} = \overset{AM-GM}{\geq} \sum \frac{4ba}{2ba \cos C} = \frac{2\sum \cos A \cos B}{\prod \cos A}$$

We have to prove, $2\sum \cos A \cos B > 9\prod \cos A$

$$\frac{2(s^2 + r^2 - 4R^2)}{4R^2} > \frac{9(s^2 - 4R^2 - 4Rr - r^2)}{4R^2}$$

$$28R^2 + 36Rr + 11r^2 > 7S^2$$

Since, $s^2 \leq 4R^2 + 4Rr + 3r^2$, then we have to prove,

$$28R^2 + 36Rr + 11r^2 > 7(4R + 4R + 3R)$$

$$8Rr > 10r^2$$

$R > \frac{10}{8} r$ ***it's true (proved)***

Solution 4 by Ravi Prakash-New Delhi-India

$$b^2 + c^2 + 2bc \geq 2bc + 2bc = 4bc$$

$b^2 + c^2 - a^2 = 2bc \cos A$, ***etc***

$$\therefore LHS \geq \frac{4bc}{2bc \cos A} + \frac{4ca}{2ac \cos B} + \frac{4ab}{2ab \cos C}$$

$= \frac{2}{\cos A} + \frac{2}{\cos B} + \frac{2}{\cos C}$ ***(1)***

For $0 < x < \frac{\pi}{2}$***, let***

$$f(x)=\frac{1}{\cos x}=\sec x$$

$$f'(x)=\sec x\tan x$$

$$f''(x)=\sec x\tan^2 x+\sec^3 x>0,\forall x\in\left(0,\frac{\pi}{2}\right)$$

Thus, $\frac{1}{3}(\sec A+\sec B+\sec C)\geq\sec\left(\frac{A+B+C}{3}\right)$

$$\Rightarrow\sec A+\sec B+\sec C\geq 6\quad (2)$$

From (1), (2): $LHS\geq 12>9$

UP.176. Let a,b be positive real numbers such that: $a+b=2$. Find the minimum value of:

$$P=\frac{1}{a^3+b^3+2}+\frac{1}{ab}+\sqrt[3]{ab}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Vietnam

$$P=\frac{1}{(a+b)[(a+b)^2-3ab]+2}+\frac{1}{ab}+\sqrt[3]{ab}=$$

$$=\frac{1}{2(4-3ab)+2}+\frac{1}{ab}+\sqrt[3]{ab}=\frac{1}{10-6ab}+\frac{1}{ab}+\sqrt[3]{ab}$$

Let $t=\sqrt[3]{ab}\Rightarrow t^3=ab$ $(0<t\leq 1$, *because:* $0<ab\leq\frac{(a+b)^2}{4}=1)$

$$P=f(t)=\frac{1}{10-6t^3}+\frac{1}{t^3}+t\Rightarrow f'(t)=1-\frac{3}{t^4}+\frac{9t^2}{2(3x^3-5)^3}<0,\forall t\in(0,1]$$

$$\Rightarrow f(t)\searrow\text{on }(0;1]\Rightarrow f(t)\geq f(1)=\frac{9}{4}\Rightarrow P_{\min}=\frac{9}{4}\Leftrightarrow a=b=1.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b>0$ *and* $a+2=2\Rightarrow ab\leq 1$. *Give* $ab=x^3\leq 1\Rightarrow x^3\leq x\leq 1$

Consider $\frac{1}{a^3+b^2+2}+\frac{1}{ab}+\sqrt[3]{ab}=\frac{1}{(a+b)^3-3ab(a+b)+2}+\frac{1}{ab}+\sqrt[3]{ab}=\frac{1}{10-6ab}+\frac{1}{ab}+\sqrt[3]{ab}\geq\frac{9}{4}$

If $\frac{1}{5-3x^3}+\frac{2}{x^3}+2x\geq\frac{9}{2}$

Iff $2x^3+20-12x^3+20x^4-12x^7\geq 45x^3-27x^6$

Iff $(12x^6 - 12x^7) + (20x^4 - 20x^3) + (15x^6 - 15x^3) + (20 - 20x^3) \geq 0$

Iff $12x^6(1-x) - 20x^3(1-x) - 15x^3(1-x^3) + 20(1-x^3) \geq 0$

Iff $12x^6 - 20x^3 - 15x^3(1+x+x^2) + 20(1+x+x^2) \geq 0$

Iff $12x^6 + 20x^2 + 20x + 20 \geq 35x^3 + 15x^4 + 15x^5$ *and it is to be true.*

Because $(10x^6 + 2) + (2x^6 + 1) \geq 12x^5 + 3x^4 \geq 15x^5$

$$20x^2 \geq 15x^4 + 5x^3$$

$$20x \geq 20x^3$$

$$10 \geq 10x^3$$

Therefore, it's minimum is $\frac{9}{4}$.

UP.177. If $x, y, z, t > 1$ then:

$$(log_{xzt}\, x)(log_{xyt}\, y)(log_{xyz}\, z)(log_{yzt}\, t) < \frac{1}{16}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

$$LHS = \frac{1}{\log_x(zxt)} \cdot \frac{1}{\log_y(xyt)} \cdot \frac{1}{\log_z(xyz)} \cdot \frac{1}{\log_t(yzt)} =$$

$$= \left(\frac{1}{1+\log_x z + \log_x t}\right) \cdot \left(\frac{1}{1+\log_y x + \log_y t}\right) \cdot \left(\frac{1}{1+\log_z x + \log_z y}\right)$$

$$\cdot \left(\frac{1}{1+\log_t y + \log_t z}\right) = \frac{1}{(1+\log_x z+\log_x t)(1+\log_y x+\log_y t)(1+\log_z x+\log_z y)(1+\log_t y+\log_t y)} \quad (*)$$

$$1 + \log_x z + \log_x t \geq 3\sqrt[3]{\log_x z \cdot \log_x t}$$

$$1 + \log_y x + \log_y t \geq 3\sqrt[3]{\log_y x \cdot \log_y t}$$

$$1 + \log_z x + \log_z y \geq 3\sqrt[3]{\log_z x \cdot \log_z y}$$

$$1 + \log_t y + \log_t z \geq 3\sqrt[3]{\log_t y \cdot \log_t z}$$

$$\Rightarrow (*) \leq \frac{1}{3^4\sqrt[3]{\log_x z \cdot \log_x t \cdot \log_y x \cdot \log_y t \cdot \log_z x \cdot \log_z y \cdot \log_t y \cdot \log_t z}} = \frac{1}{3^4} < \frac{1}{16}. \text{ Proved.}$$

Solution 2 by Amit Dutta-Jamshedpur-India

$$\because AM \geq GM$$

$$\log x+\log z+\log t\ge 3\sqrt[3]{\log x\log z\log t}\Rightarrow \log(xzt)\ge \sqrt[3]{(\log x)(\log z)(\log t)}$$

Similarly, $\log(xyt)\ge 3\sqrt[3]{(\log x)(\log y)(\log t)}$

$$\log(xyz)\ge 3\sqrt[3]{(\log x)(\log y)(\log z)}$$

$$\log(yzt)\ge 3\sqrt[3]{(\log y)(\log z)(\log t)}$$

Let $P=(\log_{xzt}x)(\log_{xyt}y)(\log_{xyz}z)(\log_{yzt}t)$

$$P=\left(\frac{\log x}{\log x+\log z+\log t}\right)\left(\frac{\log y}{\log x+\log y+\log t}\right)\left(\frac{\log z}{\log x+\log y+\log z}\right)\left(\frac{\log t}{\log y+\log z+\log t}\right)$$

$$P\overset{AM>GM}{<}\left(\frac{\log x}{3\sqrt[3]{\log x\log z\log t}}\right)\left(\frac{\log y}{3\sqrt[3]{\log x\log y\log t}}\right)\left(\frac{\log z}{3\sqrt[3]{\log x\log y\log z}}\right)\left(\frac{\log t}{3\sqrt[3]{\log y\log z\log t}}\right)$$

$P<\frac{1}{81}<\frac{1}{16}$ *(proved)*

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z,t>1$, *we have:* $(\log_{xzt}x)(\log_{ytx}y)(\log_{zxy}z)(\log_{tyz}t)=$

$$=\left(\frac{1}{1+\log_x z+\log_x t}\right)\left(\frac{1}{1+\log_y t+\log_y x}\right)\left(\frac{1}{1+\log_z x+\log_z y}\right)\left(\frac{1}{1+\log_t y+\log_t z}\right)$$

$\le\frac{1}{(1+1+1)^4}=\frac{1}{3^4}=\frac{1}{81}<\frac{1}{16}$ *Ok*

Therefore, it is to be true

$$1=(\log_x z)(\log_y x)(\log_z y)(1)$$

$$1=(\log_x t)(\log_z x)(\log_t z)(1)$$

$$1=(\log_y t)(\log_t z)(log_z t)(1)$$

UP.178. Let be $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$; $B=\begin{pmatrix}1&0\\1&1\end{pmatrix}$.

Find: $\Omega=e^A\cdot(e^B)^{-1}$; ($e^A$ – exponential matrix)

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},A^n=\begin{pmatrix}1&n\\0&1\end{pmatrix}$$

$$B=\begin{pmatrix}1&0\\1&1\end{pmatrix},B^n=\begin{pmatrix}1&0\\n&1\end{pmatrix}$$

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e & e \\ 0 & e \end{pmatrix} = e\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Similarly, $e^B = \begin{pmatrix} e & 0 \\ e & e \end{pmatrix} = e\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$\Omega = e^A(e^B)^{-1} = e\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\left(e^{-1}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

UP.179. If in ΔABC, $a \ge b \ge c$ then the following relationship holds:

$$\sqrt[5]{\frac{m_a}{m_b}} + \sqrt[5]{\frac{m_b}{m_c}} + \sqrt[5]{\frac{m_c}{m_a}} - \sqrt[5]{\frac{m_a}{m_c}} - \sqrt[5]{\frac{m_b}{m_a}} - \sqrt[5]{\frac{m_c}{m_b}} < 1$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

First, we prove that if $x \le y \le z$ then:

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \ge \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \quad (1)$$

$$\frac{x}{y} - \frac{y}{x} + \frac{y}{z} - \frac{z}{y} + \frac{z}{x} - \frac{x}{z} \ge 0$$

$$\frac{x^2 - y^2}{xy} + \frac{y^2 - z^2}{2y} + \frac{z^2 - x^2}{xz} \ge 0$$

$$z(x^2 - y^2) + x(y^2 - z^2) + y(z^2 - x^2) \ge 0$$

$$x^2z - zy^2 + xy^2 - xz^2 + yz^2 - yx^2 \ge 0$$

$$xz(x - z) + y^2(x - z) + y(z - x)(z + x) \ge 0$$

$$(x - z)(xz + y^2 - yz - yx) \ge 0$$

$$(x - z)[y(y - x) - z(y - x)] \ge 0$$

$$(x - z)(y - x)(y - z) \ge 0$$

$(z - x)(y - x)(z - y) \ge 0$ *which is true because,*

$$z - x \ge 0; y - x \ge 0, z - y \ge 0$$

By $a \le b \le c \Rightarrow m_a \ge m_b \ge m_c \Rightarrow \sqrt[5]{m_a} \le \sqrt[5]{m_b} \le \sqrt[5]{m_c}$

We take in (1):

$$x = \sqrt[5]{m_a};\ y = \sqrt[5]{m_b};\ z = \sqrt[5]{m_c}$$

$$\frac{\sqrt[5]{m_a}}{\sqrt[5]{m_b}} + \frac{\sqrt[5]{m_b}}{\sqrt[5]{m_c}} + \frac{\sqrt[5]{m_c}}{\sqrt[5]{m_a}} \ge \frac{\sqrt[5]{m_b}}{\sqrt[5]{m_a}} + \frac{\sqrt[5]{m_c}}{\sqrt[5]{m_b}} + \frac{\sqrt[5]{m_a}}{\sqrt[5]{m_c}}$$

$$\sqrt[5]{\frac{m_a}{m_b}} + \sqrt[5]{\frac{m_b}{m_c}} + \sqrt[5]{\frac{m_c}{m_a}} - \sqrt[5]{\frac{m_a}{m_c}} - \sqrt[5]{\frac{m_b}{m_a}} - \sqrt[5]{\frac{m_c}{m_b}} < 1$$

UP.180. If $f:(0,\infty) \to (0,\infty)$ such that exists

$$\lim_{x\to\infty} \frac{f(x+1)}{xf(x)} = a > 0 \text{ and exists } \lim_{x\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x} \text{ then find:}$$

$$\Omega = \lim_{x\to\infty}\left((f(x))^{\frac{2}{x+1}} \cdot \left(\frac{(f(x+1))^{\frac{1}{x+1}}}{(x+1)^2} - \frac{(f(x))^{\frac{1}{x}}}{x^2}\right)\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Pierre Mounir Cairo-Egypt

Given: $f:(0,\infty) \to (0,\infty), \lim_{x\to\infty} \frac{f(x+1)}{xf(x)} = a > 0$

Find: $\Omega = \lim_{x\to\infty} f(x)^{\frac{2}{x+1}}\left[\frac{f(x+1)^{\frac{1}{x+1}}}{(x+1)^2} - \frac{f(x)^{\frac{1}{x}}}{x^2}\right]$

We'll make use of the following two theorems of Cauchy:

(1) Let f be defined on (a,∞), and $f(x) > 0\ \forall x$ and

$\lim_{x\to\infty} \frac{f(x+1)}{f(x)}$ ***exists, then*** $\lim_{x\to\infty} f(x)^{\frac{1}{x}} = \lim_{x\to\infty} \frac{f(x+1)}{f(x)}$

(2) Let f be defined on (a,∞) and $\lim_{x\to\infty}[f(x+1) - f(x)]$

exists, then $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty}[f(x+1) - f(x)]$

Now, let $g(x) = \frac{f(x)}{x^x}$, ***then*** $g(x) > 0\ \ (x, f(x) > 0)$

$$\because \lim_{x\to\infty} \frac{g(x+1)}{g(x)} = \lim_{x\to\infty} \frac{f(x+1)}{(x+1)^{x+1}} \times \frac{x^x}{f(x)}$$

$$= \lim_{x\to\infty} \frac{f(x+1)}{xf(x)} \times \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)^{x+1}} = a \times \frac{1}{e} = \frac{a}{e}$$

∴ According to theorem (1) above:

$$\lim_{x\to\infty} g(x)^{\frac{1}{x}} = \lim_{x\to\infty}\left[\frac{f(x)}{x^x}\right]^{\frac{1}{x}} = \lim_{x\to\infty}\frac{g(x+1)}{g(x)} = \frac{a}{e}$$

$$\therefore \lim_{x\to\infty}\frac{f(x)^{\frac{1}{x}}}{x} = \frac{a}{e} = \lim_{x\to\infty}\frac{f(x+1)^{\frac{1}{x+1}}}{x+1} \quad (x \to x+1)$$

Also, let $h(x) = \frac{f(x)^{\frac{1}{x}}}{x^2}$***, then:***

$$\because \lim_{x\to\infty}[h(x+1)-h(x)] = \lim_{x\to\infty}\left[\frac{f(x+1)^{\frac{1}{x+1}}}{(x+1)^2} - \frac{f(x)^{\frac{1}{x}}}{x^2}\right] =$$

$$\lim_{x\to\infty}\frac{1}{x+1} \times \frac{f(x+1)^{\frac{1}{x+1}}}{x+1} - \lim_{x\to\infty}\frac{1}{x} \times \frac{f(x)^{\frac{1}{x}}}{x} = 0 \times \frac{a}{e} - 0 \times \frac{a}{e} = 0$$

∴ According to theorem (2) above:

$$\lim_{x\to\infty}\frac{h(x)}{x} = \lim_{x\to\infty}[h(x+1)-h(x)] = \lim_{x\to\infty}\frac{f(x)^{\frac{1}{x}}}{x^3} \to (*)$$

$$\therefore \Omega = \lim_{x\to\infty} f(x)^{\frac{2}{x+1}}\left[\frac{f(x+1)^{\frac{1}{x+1}}}{(x+1)^2} - \frac{f(x)^{\frac{1}{x}}}{x^2}\right]$$

$$= \lim_{x\to\infty} f(x)^{\frac{2}{x+1}} \times \lim_{x\to\infty}[h(x+1)-h(x)]$$

$$= \lim_{x\to\infty} f(x)^{\frac{2}{x+1}} \times \lim_{x\to\infty}\frac{f(x)^{\frac{1}{x}}}{x^3} \quad \textit{[from (*)]}$$

$$= \lim_{x\to\infty} f(x)^{\frac{2}{x+1}+\frac{1}{x}} \times \frac{1}{x^3} = \lim_{x\to\infty}\frac{f(x)^{\frac{3x+1}{x(x+1)}}}{(x)^{\frac{3x+1}{x+1}}} \times \frac{(x)^{\frac{3x+1}{x+1}}}{x^3}$$

$$= \lim_{x\to\infty}\left[\frac{f(x)^{\frac{1}{x}}}{x}\right]^{\frac{3x+1}{x+1}} \times \left(x^{\frac{1}{x}}\right)^{-\frac{2x}{x+1}}$$

$$= \left[\lim_{x\to\infty}\frac{f(x)^{\frac{1}{x}}}{x}\right]^{\lim\limits_{x\to\infty}\frac{3x+1}{x+1}} \times \left(\lim_{x\to\infty} x^{\frac{1}{x}}\right)^{\lim\limits_{x\to\infty}-\frac{2x}{x+1}}$$

$$= \left(\frac{a}{e}\right)^3 \times (1)^{-2} = \left(\frac{a}{e}\right)^3$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\Omega = \lim_{x\to\infty}\left(f(x)\right)^{\frac{2}{x+1}}\left(\frac{\left(f(x+1)\right)^{\frac{1}{x+1}}}{(x+1)^2} - \frac{\left(f(x)\right)^{\frac{1}{x}}}{x^2}\right) =$$

$$= \lim_{x\to\infty}\left(x^{-\frac{2}{x+1}}\left(\frac{\left(f(x)\right)^{\frac{1}{x}}}{x}\right)^{\frac{2}{1+\frac{1}{x}}}\left(x^2\left(\frac{\left(\frac{\left(f(x+1)\right)^{\frac{1}{x+1}}}{x+1}\right)}{x+1} - \frac{\left(\frac{\left(f(x)\right)^{\frac{1}{x}}}{x}\right)}{x}\right)\right)\right)$$

$$= \lim_{x\to\infty}\left(\left(\frac{\frac{f(x+1)}{(x+1)^{x+1}}}{\frac{f(x)}{x^x}}\right)^2\left(-\frac{\frac{f(x+1)}{(x+1)^{x+1}}}{\frac{f(x)}{x^x}}\right)\right) = -\lim_{x\to\infty}\left(\frac{\frac{f(x+1)}{xf(x)}}{\left(1+\frac{1}{x}\right)^{x+1}}\right)^3$$

$$\therefore \lim_{x\to\infty}\left(f(x)\right)^{\frac{2}{x+1}}\left(\frac{\left(f(x+1)\right)^{\frac{1}{x+1}}}{(x+1)^2} - \frac{\left(f(x)\right)^{\frac{1}{x}}}{x^2}\right) = \left(\frac{a}{e}\right)^3$$