Number 13 *SUMMER 2019*

R M M

ROMANIAN MATHEMATICAL MAGAZINE

SOLUTIONS

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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JP.181. Let x, y, z be positive real numbers such that $x+y+z=3$. Find the minimum value of:

$$T=x^2\cdot\sqrt{\frac{3yz}{2y^2-yz+2z^2}}+y^2\cdot\sqrt{\frac{3zx}{2z^2-zx+2x^2}}+$$
$$+z^2\cdot\sqrt{\frac{3xy}{2x^2-xy+2y^2}}+\frac{27}{xy+yz+zx}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Michael Sterghiou-Greece

$T=\left(\sum_{cyc}x^2\sqrt{\frac{3yz}{2y^2-yz+2z^2}}\right)+\frac{27}{\sum_{cyc}xy}$ *(1) Let* $(p,q,r)=\left(\sum_{cyc}x,\sum_{cyc}xy,xyz\right), p=3$. *We will show that* $T\geq 12$. *If* $\frac{27}{\sum_{cyc}xy}=\frac{27}{q}\geq 12\to q\leq\frac{9}{4}$ *we are done as the rest of* T *is positive, so we assume* $\frac{9}{4}\leq q\leq 3$.

(1)$\to T=\left(\sum_{cyc}x^2\cdot\frac{\sqrt{3xyz}}{\sqrt{2xy^2-xyz+2xz^2}}\right)+\frac{27}{q}\geq 12$ *or* $\sqrt{3r}\cdot\sum_{cyc}\frac{x^2}{\sqrt{2xy^2-r+2xz^2}}+\frac{27}{q}\geq 12$

The function $f(t)=\frac{1}{\sqrt{t}}$ *is convex on* $(0,3)$ $\left[f''(t)=\frac{3}{4t^{\frac{5}{2}}}>0\right]$

Applying the generalized Jensen with $a_i=x^2,y^2,z^2$ *for* $i=1,2,3$ *we have*

$T\geq\sqrt{3r}\cdot\sum_{cyc}x^2\cdot\frac{1}{\sqrt{\frac{\sum_{cyc}x^2(2xy^2-r+2xz^2)}{\sum_{cyc}x^2}}}+\frac{27}{q}\geq 12$ *(2) Given that* $12\cdot\frac{27}{q}\geq 0$ *and*

$\sum_{cyc}x^3y^2+\sum_{cyc}x^3z^2=\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)-\left(\sum_{cyc}xy\right)\cdot xyz=3(q^2-6r)-qr$ *we arrive at:* $\frac{3r(9-2q)^2}{6(q^2-6r)-2qr-(9-2q)r}\geq\left(12-\frac{27}{q}\right)^2$ *or*

$$f(q)=r\left[-24q^5+324q^4-1170q^3+6075q^2-21870q+26244\right]-864q^4+$$
$$+3600q^3-1782q^2-7290q+6561$$

The function in the brackets $g(q)$ can easily show to be ≥ 0 (starting from 3rd derivative $g'''(q) = -36(40q^2 - 216q + 195) > 0$ for $\frac{9}{4} \leq q \leq 3$ and going up). So, we can obtain the stronger inequality replacing $r \geq \frac{4q-9}{3}$ (Schur):

$$h(q) = \overbrace{(4q-9)}^{\geq 0}\overbrace{(3-q)}^{\geq 0}\,(8q^4 - 84q^3 + 354q^2 - 1377q + 2673)$$

$= (4q-9)(3-q)\cdot\gamma(q)$: *But* $\gamma''(q) = 12(8q^2 - 42q + 59) > 0 \to \gamma'(q) \uparrow \leq \gamma'(3) < 0$ $\to \gamma(q) \downarrow \to \gamma(q) > \gamma(3) > 0$ *and hence* $h(q) \geq 0$. *Equality for* $x = y = z = 1$

Done!

JP.182. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that:

$$\left(a^5 - 2a + 4\right)\left(b^5 - 2b + 4\right)\left(c^5 - 2c + 4\right) \geq 9\sqrt{3(a^2 + b^2 + c^2)}$$

Equality occurs if and only if?

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have: $a^5 - 2a + 4 \geq a^3 + 2 \Leftrightarrow a^5 - 2a - a^3 + 2 \geq 0$

$\Leftrightarrow (a-1)^2(a^3 + 2a^2 + 2a + 2) \geq 0$ *(true for* $a > 0$*)*

Similarly: $b^5 - 2b + 4 \geq b^3 + 2; c^5 - 2c + 4 \geq c^3 + 2;$

By Hölder's inequality, we have:

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) = (a^3 + 1^3 + 1^3)(1^3 + b^3 + 1^3)(1^3 + 1^3 + c^3)$$
$$\geq (a + b + c)^3$$

Must show that: $(a + b + c)^3 \geq 9\sqrt{3(a^2 + b^2 + c^2)}$

$$\Leftrightarrow t^3 \geq 9\sqrt{3(t^2 - 6)} \quad \left(t = a + b + c \geq \sqrt{3(ab + bc + ac)} = 3\right)$$

$$\Leftrightarrow t^6 \geq \left(9\sqrt{3(t^2 - 6)}\right)^2 \Leftrightarrow t^6 - 243t^2 + 1458 \geq 0 \Leftrightarrow (t-3)^2(t+3)^2(t^2 + 18) \geq 0$$

(True)

Equality $\Leftrightarrow a = b = c = 1.$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For all $x > 0$***, we get that:***

$$x^5 - 2x + 4 = (x^5 + 1) + 3 - 2x = (x + 1)(x^4 - x^3 + x^2 - x + 1) + 3 - 2x$$

$$\geq (x + 1)(x^3 - x + 1) + 3 - 2x = x^4 + x^3 - x^2 - 2x + 4$$

$$\geq x^3 + x^2 - 2x + 3 \geq x^3 + 2: \textit{fact}$$

Hence for $a, b, c > 0$ ***and*** $ab + bc + ca = 3$***, we have***

$$(a^5 - 2a + 4)(b^5 - 2b + 4)(c^5 - 2c + 4) \geq 9\sqrt{3(a^2 + b^2 + c^2)}$$

$$\textit{If } (a^3 + 2)(b^3 + 2)(c^2 + 2) \geq 9\sqrt{3(a^2 + b^2 + c^2)}$$

$$\textit{If } (a + b + c)^3 \geq 9\sqrt{3(a^2 + b^2 + c^2)}$$

$$\textit{If } (a + b + c)^6 \geq 243(a^2 + b^2 + c^2)$$

$$\textit{If } \left(a^2 + b^2 + c^2 + 2(ab + bc + ca)\right)^3 \geq 243(a^2 + b^2 + c^2)$$

$$\textit{If } (a^2 + b^2 + c^2 + 6)^3 \geq 243(a^2 + b^2 + c^2)$$

$$\textit{If } a^2 + b^2 + c^2 + 6 \geq 3\sqrt[3]{9(a^2 + b^2 + c^2)}$$

$$\textit{If } a^2 + b^2 + c^2 + 6 \geq 3\sqrt[3]{\left(\frac{243 + a^2 + b^2 + c^2}{3}\right)^3} = \frac{3(6 + a^2 + b^2 + c^2)}{3} = a^2 + b^2 + c^2 + 6 \ \textit{ok}$$

Therefore, it's true.

JP.183. In ΔABC the following relationship holds:

$$3s\left(\frac{2r}{R}\right)^2 \leq \sum h_a^2\left(\frac{1}{b} + \frac{1}{c}\right) \leq 3s$$

Proposed by Marin Chirciu – Romania

Solution 1 by Mustafa Tarek-Cairo-Egypt

$$\sum h_a^2\left(\frac{1}{b} + \frac{1}{c}\right) = c\frac{4\Delta^2}{\Delta^2}\left(\frac{b+c}{bc}\right) = \frac{4\Delta^2}{abc}\sum\left(\frac{b+c}{a}\right) = \frac{sr}{R}\sum\left(\frac{b+c}{a}\right) \quad \textit{(a)}$$

$$\textit{(1)} \Leftrightarrow \sum\left(\frac{b+c}{a}\right) \geq 12\frac{r}{R}, \textit{ but } \frac{b}{a} + \frac{a}{b} \geq 2 \quad \textit{etc}$$

$$\therefore \sum\left(\frac{b+c}{a}\right) = \sum\left(\frac{b}{a} + \frac{a}{b}\right) \geq 6 \overset{??}{\geq} 12\frac{r}{R} \Rightarrow \textit{true } \frac{1}{2} \geq \frac{r}{R}$$

$$\textit{(1) (proved), we have } \frac{h_a}{w_a} = \frac{b+c}{a}\sin\frac{A}{2} \leq 1 \quad \textit{etc}$$

$$\therefore \textit{(3)} = \frac{sr}{R}\sum\left(\frac{b+c}{a}\right) \le \frac{sr}{R}\sum\sin\frac{A}{2} = \frac{S}{R}\sum AI \overset{??}{\le} 3s$$

We must prove $\sum AI \le 3R$***, to prove that we will prove that*** $\sum AI \overset{(1)}{\le} \sqrt{\sum ab} \overset{(5)}{\le} 2$

$$\textit{(5)} \Leftrightarrow s^2 + 4Rr + r^2 \le 4R^2 + 4r^2 + 8Rr \leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2$$

True → (Gerretsen)→ $(5$***-proved)***

$$\therefore \sum AI \le 2(R+r) \le 3R \leftrightarrow 2r \le R \rightarrow \textit{true (Euler)}$$

$$\therefore \sum AI \le 3R, \therefore 2 \quad \textit{(proved)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Using $h_a = \frac{2S}{a}$ ***we obtain:*** $\sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right) = \sum\frac{b+c}{bc}\cdot\frac{4S^2}{a^2} = \frac{4S^2}{abc}\sum\frac{b+c}{a}$

$$= \frac{4S^2}{4RS}\sum\frac{b+c}{a} = \frac{S}{R}\sum\frac{b+c}{a} = \Omega$$

$$\because \sum\frac{2s-a}{a} = 2s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 3$$

$$= 2s\left(\frac{ab+bc+ca}{abc}\right) - 3 = 2s\left(\frac{s^2+4Rr+r^2}{4Rsr}\right) - 3$$

$$= \frac{s^2+4Rr+r^2}{2Rr} - 3 = \frac{s^2-2Rr+r^2}{2Rr}$$

$$\Rightarrow \Omega = \frac{sr}{R}\cdot\frac{s^2-2Rr+r^2}{2Rr} = \frac{s}{2R^2}(s^2-2Rr+r^2)$$

$$\Omega \ge 3s\left(\frac{2r}{R}\right)^2 \quad \textit{(1)}$$

$$\Leftrightarrow s^2 - 2Rr + r^2 \ge 24r^2 \Leftrightarrow s^2 - 2Rr \ge 23r^2$$

$$\therefore s^2 \ge 16Rr - 5r^2 \Rightarrow 14Rr - 5r^2 \ge 23r^2$$

$$\Leftrightarrow 14Rr \ge 28r^2 \Leftrightarrow R \ge 2r \textit{ (true)} \Rightarrow \textit{(1) true.}$$

$$\Omega \le 3s \quad \textit{(2)}$$

$$\Leftrightarrow s^2 - 2Rr + r^2 \le 6R^2$$

$$\therefore s^2 \le 4R^2 + 4Rr + 3r^2 \Rightarrow 4R^2 + 2Rr + 4r^2 \le 6R^2$$

$$\Leftrightarrow 2Rr + 4r^2 \le 2R^2 \Leftrightarrow Rr + 2r^2 \le R^2$$

$$\textit{(True} \because Rr + 2r^2 \overset{(Euler)}{\le} R\cdot\frac{R}{2} + 2\cdot\frac{R^2}{4} = R^2\textit{)} \Rightarrow \textit{(2) true.}$$

JP.184. In ΔABC the following relationship holds:

$$\frac{18r}{R} \le \sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 \le 9$$

Proposed by Marin Chirciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 = \sum \frac{4S^2}{a^2}\left(\frac{b+c}{bc}\right)^2 = \sum \frac{4S^2(b+c)^2}{a^2b^2c^2} = \sum \frac{(b+c)^2}{4R^2} \quad (1)$$

From (1) we must show: $\frac{18r}{R} \le \frac{\sum(b+c)^2}{4R^2} \le 9 \Leftrightarrow 72Rr \le \sum(b+c)^2 \le 36R^2$ ***(2)***

$$\sum(b+c)^2 = 2(a^2+b^2+c^2+ab+ac+bc) \quad (3)$$

But $ab+ac+bc \le a^2+b^2+c^2 \le 9R^2$ ***(4)***

From (3)+ (4)$\Rightarrow \sum(b+c)^2 \le 36R^2$ ***(5)***

From Cauchy's inequality $\Rightarrow \sum(b+c)^2 \ge \frac{1}{3}\left(\sum(b+c)\right)^2 \Leftrightarrow \sum(b+c)^2 \ge \frac{16s^2}{3} \Rightarrow$ *we must show:* $\frac{16s^2}{3} \ge 72Rr \Leftrightarrow 2s^2 \ge 27Rr$ *which it is true (6)*

From (4)+ (6) $\Rightarrow$ *2 it is true.*

Solution 2 by Tran Hong-Dong Thap-Vietnam

Using $h_a = \frac{2\Delta}{a}$ *(etc) we obtain:* $\Omega = \sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 = \sum \frac{4\Delta^2}{a^2}\cdot\frac{(b+c)^2}{(bc)^2} = \frac{4\Delta^2}{(abc)^2}\sum(b+c)^2$

$$= \frac{2(a^2+b^2+c^2+ab+bc+ca)}{4R^2} = \frac{a^2+b^2+c^2+ab+bc+ca}{2R^2}$$

$$= \frac{2s^2-8Rr-2r^2+s^2+4Rr+r^2}{2R^2} = \frac{3s^2-4Rr-r^2}{2R^2}$$

$$\Omega \ge \frac{18r}{R} \Leftrightarrow 3s^2-4Rr-r^2 \ge 36Rr \Leftrightarrow 3s^2 \ge 40Rr+r^2$$

$\because s^2 \ge 16Rr-5r^2 \Rightarrow 48Rr-15r^2 \ge 40Rr+r^2 \Leftrightarrow 8Rr \ge 16r^2 \Leftrightarrow R \ge 2r$ ***(Euler)***

$$\Omega \le 9 \Leftrightarrow 3s^2-4Rr-r^2 \le 18R^2 \Leftrightarrow 3s^2 \le 18R^2+4Rr+r^2$$

$$\therefore s^2 \le 4R^2+3r^2+4Rr \Rightarrow 12R^2+9r^2+12Rr \le 18R^2+4Rr+r^2$$

$$\Leftrightarrow 8Rr+8r^2 \le 6R^2 \Leftrightarrow 4Rr+4r^2 \le 3R^2$$

It is true because: $4Rr+4r^2 \overset{(Euler)}{\le} 4R\cdot\frac{R}{2}+4\cdot\frac{R^2}{4} = 3R^2$*. Proved.*

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{18r}{R} \overset{(1)}{\leq} \sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 \overset{(2)}{\leq} 9$$

$$\sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 = \sum \frac{b^2c^2}{4R^2}\cdot\frac{(b+c)^2}{b^2c^2}$$

$$= \frac{1}{4R^2}\sum(b+c)^2 \overset{Bogdan\ Fustei}{\leq} \frac{1}{4R^2}\sum\left(2\sqrt{R(r_b+r_c)}\right)^2$$

$$= \frac{1}{4R^2}\sum\left(4R\cdot 4R\cos^2\frac{A}{2}\right) = 2\sum(1+\cos A)$$

$$= 2\left(\frac{4R+r}{R}\right) \overset{Euler}{\leq} \frac{8R+r}{R} = 9 \Rightarrow \textbf{(2) is true}$$

$$\text{Also, } \sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 = \frac{1}{4R^2}\sum(b+c)^2 \geq \frac{1}{12R^2}\left\{\sum(b+c)\right\}^2 = \frac{16S^2}{12R^2} = \frac{4s^2}{3R^2} \overset{?}{\geq} \frac{18r}{R}$$

$$\Leftrightarrow 2s^2 \overset{?}{\geq} 27Rr \Leftrightarrow 2(s^2-16Rr+5r^2)+5r(R-2r) \overset{?}{\geq} 0$$

→ ***true by Gerretsen & Euler*** ⇒ ***(1) is true (proved)***

JP.185. In ΔABC the following relationship holds:

$$\sum r_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 \geq \frac{9R}{2r}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{r^2S^2}{(s-a)^2}\cdot\frac{(b+c)^2a^2}{16R^2r^2s^2} \overset{(1)}{=} \frac{1}{16R^2}\sum\frac{a^2(b+c)^2}{(s-a)^2}$$

$$\text{Now, } \sum\frac{a^2(b+c)^2}{(s-a)^2} = \sum\frac{a^2(s+s-a)^2}{(s-a)^2} = \sum\frac{a^2\{s^2+(s-a)^2+2s(s-a)\}}{(s-a)^2}$$

$$= s^2\sum\frac{a^2-s^2+s^2}{(s-a)^2} + 2(s^2-4Rr-r^2) + 2s\sum\frac{a^2-s^2+s^2}{(s-a)}$$

$$= s^2\sum\frac{a+s}{a-s} + \frac{s^4}{s^2r^2}\left(\sum\frac{r^2s^2}{(s-a)^2}\right) + 2(s^2-4Rr-r^2) -$$

$$-2s\sum(a+s) + \frac{2s^3}{r^2s}\sum(s-b)(s-c)$$

$$= s^2\sum\frac{a-s+2s}{a-s}+\frac{s^2}{r^2}\left(\sum r_a^2\right)+2(s^2-4Rr-r^2)-2s(5s)+$$

$$+\frac{2s^2}{r^2}\sum(s^2-s(b+c)+bc)$$

$$= s^2(3)-\frac{2s^3}{r^2s}\sum(s^2-s(b+c)+bc)+\frac{s^2}{r^2}\{(4R+r)^2-2s^2\}+$$

$$+2(s^2-4Rr-r^2)-10s^2+\frac{2s^2}{r^2}\sum(s^2-s(b+c)+bc)$$

$$= -5s^2-8Rr-2r^2+\frac{s^2}{r^2}\{(4R+r)^2-2s^2\}$$

$$= \frac{-r^2(5s^2+8Rr+2r^2)+s^2(16R^2+8Rr+r^2)-2s^4}{r^2}$$

$$\overset{(2)}{=}\frac{s^2(16R^2+8Rr-4r^2)-2r^3(4R+r)-2s^4}{r^2}$$

(1), (2) $\Rightarrow LHS=\frac{s^2(8R^2+4Rr-2r^2)-r^3(4R+r)-s^4}{8R^2r^2}\ge\frac{9R}{2r}$

$$\Leftrightarrow\frac{s^2(8R^2+4Rr-2r^2)-r^3(4R+r)-s^4-36R^3r}{8R^2r^2}\ge 0$$

$$\Leftrightarrow s^2(8R^2+4Rr-2r^2)\overset{(3)}{\ge} s^4+r^3(4R+r)+36R^3r$$

Now, RHS of (3) $\overset{Gerretsen}{\le} s^2(4R^2+4Rr+3r^2)+r^3(4R+r)+36R^3r$

$$\overset{?}{\le} s^2(8R^2+4Rr-2r^2)\Leftrightarrow s^2(4R^2-5r^2)-r^3(4R+r)-36R^3r\underset{(4)}{\overset{?}{\ge}}0$$

Again, LHS of (4) $\overset{Gerretsen}{\ge} r(16R-5r)(4R^2-5r^2)-r^3(4R+r)-36R^3r\overset{?}{\ge}0$

$$\Leftrightarrow 7t^3-5t^2-21t+6\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(7t^2+9(t-2)+15)\overset{?}{\ge}0$$

$\rightarrow$ ***true*** $\because t\overset{Euler}{\ge}2\Rightarrow$ ***(4)*** $\Rightarrow$ ***(3)*** $\Rightarrow$ ***given inequality is true (proved)***

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sum r_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2=\sum\frac{\Delta^2}{(s-a)^2}\cdot\frac{(b+c)^2}{(bc)^2}=\frac{1}{(4R)^2}\sum\left[\frac{a(b+c)}{s-a}\right]^2=\Omega$$

$$\sum\left[\frac{a(b+c)}{s-a}\right]^2\ge\frac{1}{3}\left[\sum\frac{a(b+c)}{s-a}\right]^2=\frac{1}{3}\left[\sum\frac{a(s+s-a)}{s-a}\right]^2$$

$$=\frac{1}{3}\left[\sum\frac{as}{s-a}+\sum a\right]^2=\frac{1}{3}\left[s\sum\frac{a}{s-a}+2s\right]^2$$

$$=\frac{1}{3}\left[s\left(\frac{4R}{r}-2\right)+2s\right]^2=\frac{1}{3}\cdot\frac{16s^2R^2}{r^2}\Rightarrow\Omega\ge\frac{s^2}{3r^2}\overset{(1)}{\ge}\frac{9R}{2r}$$

(1) $\Leftrightarrow 2s^2\ge 27Rr$

$$\therefore s^2\ge 16Rr-5r^2\Rightarrow 2s^2\ge 32Rr-10r^2$$

$\Rightarrow 32Rr-10r^2\ge 27Rr\Leftrightarrow 5Rr\ge 10r^2\Leftrightarrow R\ge 2r$ ***(Euler)*** $\Rightarrow$ ***(1) true. Proved.***

JP.186. Solve for real numbers:

$$\begin{cases}2x^2+y^2=x\sqrt{y}(2\sqrt{x}+\sqrt{y})\\x^5-3\sqrt{xy}+4\le\sqrt{2y^2-2x+1}+\sqrt[3]{3x^3-3xy+1}\end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

$x,y\ge 0$; ***let*** $y=tx\quad(t\ge 0)$

$$2x^2+t^2x^2=x\sqrt{tx}(2\sqrt{x}+\sqrt{tx})\Leftrightarrow x^2(2+t^2)=x^2\sqrt{t}(2+\sqrt{t})$$

$$\Leftrightarrow\left[\begin{array}{l}x^2=0\\2+t^2-(2\sqrt{t}+t)=0\end{array}\right.\Leftrightarrow\left[\begin{array}{l}x=0\\t^2-(t+2\sqrt{t})+2=0\quad(*)\end{array}\right.$$

$\therefore$ ***If*** $x=0\Rightarrow y=0$ ***then:***

$$0^5-3\sqrt{0\cdot 0}+4\le\sqrt{2\cdot 0^2-2\cdot 0+1}+\sqrt[3]{3\cdot 0^3-3\cdot 0\cdot 0+1}$$

$\Leftrightarrow 4\le 2$ ***(contrary)*** $\because t^2-(t+2\sqrt{t})+2=0$. ***Let*** $u=\sqrt{t}\ (u\ge 0)$

$$u^4-(u^2+2u)+2=0$$

$$\Leftrightarrow u^4-u^2-2u+2=0\Leftrightarrow(u-1)^2[(u+1)^2+1]=0$$

$$\Leftrightarrow u=1\Leftrightarrow t=1\Leftrightarrow y=x$$

$\Rightarrow x^5-3x+4\le\sqrt{2x^2-2x+1}+\sqrt[3]{3x^3-3x^2+1}$ ***(1)***

We must show that: $\sqrt{2x^2-2x+1}+\sqrt[3]{3x^3-3x^2+1}\le x^5-3x+4$ ***(2)***

$x^5-3x+4\ge 2\sqrt{2x^2-2x+1}$ ***(3)***

$$\Leftrightarrow(x^5-3x+4)^2\ge 4(2x^2-2x+1)$$

$$\Leftrightarrow(x-1)^2(x^8+2x^7+3x^6+4x^5-x^4+2x^3+5x^2+8x+12)\ge 0$$

It is true with $x \ge 0$***:***

$$\because 0 \le x \le 1 \Rightarrow x^8 + 2x^7 + 3x^6 + 4x^5 + 2x^3 + 5x^2 + 8x + 12 - x^4 \ge 12 - 1 = 11 > 0$$

$$\because x > 1 \Rightarrow x^4(x^2 - 1) + 2x^7 + 3x^6 + 4x^5 + 2x^3 + 5x^2 + 8x + 12 > 0$$

$$x^5 - 3x + 4 \ge 2\sqrt[3]{3x^3 - 3x^2 + 1} \quad (4)$$

$$\Leftrightarrow \left(x^5 - 3x + 4\right)^3 \ge 8(3x^3 - 3x^2 + 1)$$

$$\Leftrightarrow (x-1)^2(x^{13} + 2x^{12} + 3x^{11} + 4x^{10} - 4x^9 +$$

$$+4x^7 + 8x^6 + 38x^5 - 2x^4 + 5x^3 + 12x^2 - 32x + 56) \ge 0$$

It is true with $x \ge 0$***:***

$$\because 0 \le x \le 1 \Rightarrow x^{13} + 2x^{12} + 3x^{11} + 4x^{10} + 8x^5 + 5x^3 + 12x^2 + 56 -$$

$$-(4x^9 + 2x^4 - 32x) > 56 - (4 + 2 + 32) = 12 > 0$$

$$\because x > 1 \Rightarrow 4x^9(x-1) > 0; 2x^4(x-1) > 0$$

$$56 + \left\{x^{13} + 2x^{12} + 3x^{11} + 6x^5 + 5x^3 + 12x^2 - 32x\right\} =$$

$$= 56 + x(x^{12} + 2x^{11} + 3x^{10} + 6x^4 + 5x^2 + 12x - 32)$$

$$> 56 + (1 + 2 + 3 + 6 + 5 + 12 - 32) = 53 > 0$$

From (3) and (4) $\Rightarrow$ ***(2) true.***

From (1) and (2) we have equality $\Leftrightarrow x = 1 \Rightarrow y = x = 1$***. Hence:*** $(x, y) = (1, 1)$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\begin{cases} 2x^2 + y^2 = x\sqrt{y}(2\sqrt{x} + \sqrt{y}) & (I) \\ x^5 - 3\sqrt{xy} + 4 \le \sqrt{2y^2 - 2x + 1} + \sqrt[3]{3x^3 - 3xy + 1} & (II) \end{cases}$$

From equation (I): $y^2 + 2x^2 = 2x\sqrt{x}\sqrt{y} + xy$

Let be the function: $f(y) = y^2 - xy - 2x\sqrt{x} \cdot \sqrt{y} + 2x^2, y \ge 0$

$$f'(y) = 2y - x - 2x\sqrt{x} \cdot \frac{1}{2\sqrt{y}};\ f'(y) = 2y - x + \frac{x\sqrt{x}}{\sqrt{y}};\ f'(y) = \frac{2y\sqrt{y} - x\sqrt{y} + x\sqrt{x}}{\sqrt{y}}$$

$$f'(y) = 0 \Rightarrow 2y\sqrt{y} - x\sqrt{y} + x\sqrt{x} = 0;\ (2y - x)\sqrt{y} = -x\sqrt{x};\ (2y - x)^2 y = x^3$$

$$4y^3 - 4y^2x + x^2 \cdot y = x^3;\ 4y^3 - 4y^2x + x^2y - x^3 = 0;\ (y - x)(4y^2 + x^2) = 0$$

$$\begin{cases} y = x \\ x = y = 0\ impossible \end{cases}$$

$$D_{f'} =]0, +\infty[$$

Equation (I) satisfying when $x=y$: $2x^2+x^2=x\sqrt{x}(2\sqrt{x}+\sqrt{x})$

$$3x^2=x\sqrt{x}(3\sqrt{x});\ 3x^2=3x^2$$

y	0	x	$+\infty$
$f'(y)$	$----------$	0	$+++++++++$
$f(y)$	$2x^2$	0	$+\infty$

For equation (II), $y=x$

$$x^5-3x+4\le\sqrt{2x^2-2x+1}+\sqrt[3]{3x^3-3x^2+1}$$

$$x^5-3x+4\le\sqrt{2x(x-1)+1}+\sqrt[3]{3x^2(x-1)+1}$$

This inequality is true if and only if: $x=1$

$$2\le\sqrt{1}+\sqrt{1},2\le 2$$

So: the common solving is $\{(x,y)=(1,1)\}$

JP.187. There is a positive integer n of $2018's$ digits such that the sequence:

$$S(S(3n)),S(S(2n)),S\left(S(S(n))\right)$$

is an increasing arithmetic progression formed by prime numbers?

Obs.: $S(n)$ denotes sum of the digits of n.

Proposed by Pedro H.O. Pantoja – Natal/RN – Brazil

Solution by proposer

Yes. We will show that the number $n=2\underbrace{3\cdots3}_{2016}5$ *satisfies the conditions of the problem. First, we calculate:* $2n=4\underbrace{6\cdots6}_{2015}70$ *and* $3n=7\underbrace{0\cdots0}_{2016}5$.

Therefore, $S(3n)=12\Rightarrow$

$S(S(3n))=3, S(2n)=11+6\cdot2015=12101\Rightarrow S(S(2n))=5$ *and finally, we get* $S(n)=7+3\cdot2016=6055\Rightarrow S(S(n))=16\Rightarrow S\left(S(S(n))\right)=7$. *Thus, the sequence* $S(S(3n)),S(S(2n)),S\left(S(S(n))\right)$ *it's the same as* $3,5,7$.

JP.188. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum value of:

$$P = \frac{x^3}{y(2y^2 - yz + 2z^2)^2} + \frac{y^3}{z(2z^2 - zx + 2x^2)^2} + \frac{z^3}{x(2x^2 - xy + 2y^2)^2} + \frac{\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}}{27}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

- By Cauchy – Schwarz inequality we have:

$$\sum \frac{x^3}{y(2y^2-yz+2z^2)^2} = \sum \frac{\left(\frac{x^2}{2y^2-yz+2z^2}\right)^2}{xy} \geq \frac{\left(\sum \frac{x^2}{2y^2-yz+2z^2}\right)^2}{\sum xy} \quad (1)$$

$$+ \text{ Other}, \sum \frac{x^2}{2y^2-yz+2z^2} = \sum \frac{x^4}{2x^2y^2-x^2yz+2x^2z^2} \geq \frac{\left(\sum x^2\right)^2}{\sum(2x^2y^2-x^2yz+2x^2z^2)} \geq 1$$

$$\Leftrightarrow \left(\sum x^2\right)^2 \geq 4\sum x^2y^2 - xyz\sum x \Leftrightarrow \sum x^4 + xyz\sum x \geq 2\sum x^2y^2 \quad (2)$$

+ By Schur's and AM-GM inequality:

$$\sum x^2(x-y)(x-z) \geq 0 \Rightarrow \sum x^4 + xyz\sum x \geq \sum xy(x^2+y^2)$$

$$\sum xy(x^2+y^2) \geq \sum xy \cdot 2xy = 2\sum x^2y^2 \Rightarrow \sum x^4 + xyz\sum x \geq 2\sum x^2y^2 \Rightarrow \textit{(3) True.}$$

$$- \text{ Let (1), (2)}: \Rightarrow \sum \frac{x^3}{y(2y^2-yz+2z^2)^2} \geq \frac{1}{\sum xy} \quad (3)$$

– By AM-GM inequality for 6 positive real numbers we have:

$$\begin{cases} \sqrt[4]{x} + \sqrt[4]{x} + \sqrt[4]{x} + \sqrt[4]{x} + x^3 + x^2 \geq 6 \cdot \sqrt[6]{x \cdot x^3 \cdot x^2} = 6x \\ \sqrt[4]{y} + \sqrt[4]{y} + \sqrt[4]{y} + \sqrt[4]{y} + y^3 + y^2 \geq 6 \cdot \sqrt[6]{y \cdot y^3 \cdot y^2} = 6y \\ \sqrt[4]{z} + \sqrt[4]{z} + \sqrt[4]{z} + \sqrt[4]{z} + z^3 + z^2 \geq 6 \cdot \sqrt[6]{z \cdot z^3 \cdot z^2} = 6z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 4 \cdot \sqrt[4]{x} \geq 6x - x^2 - x^3 \\ 4 \cdot \sqrt[4]{y} \geq 6y - y^2 - y^3 \\ 4 \cdot \sqrt[4]{z} \geq 6z - z^2 - z^3 \end{cases}$$

$$\Rightarrow 4\left(\sum \sqrt[4]{x}\right) \geq 6\sum x - \sum x^2 - \sum x^3 = 6 \cdot 3 - \left(\sum x\right)^2 + 2\sum xy - \sum x^3 = 2\sum xy + 9 - \sum x^3 \quad (4)$$

+ Other, because

$$x+y+z=3; x,y,z>0 \Rightarrow \sum (x-3)(x-1)^2 \le 0 \Leftrightarrow \sum (x-3)(x^2-2x+1) \le 0$$

$$\Leftrightarrow \sum x^3 - 5\sum x^2 + 7\sum x - 9 \le 0 \Leftrightarrow \sum x^3 \le 5\sum x^2 - 7\sum x + 9 = 5\cdot 3^2 - 10\sum xy - 7\cdot 3 + 9$$

$\Leftrightarrow \sum x^3 \le 33 - 10\sum xy$. ***Let (4):*** $\Rightarrow 4\left(\sum \sqrt[4]{x}\right) \ge 2\sum xy + 9 - (33 - 10\sum xy) \Leftrightarrow \sum \sqrt[4]{x} \ge 3\sum xy - 6$ ***(5)***

– Let (4), (5) and using AM-GM inequality:

$$\Rightarrow P \ge \frac{1}{\sum xy} + \frac{3\sum xy - 6}{27} = \frac{1}{\sum xy} + \sum xy - \frac{2}{9} \ge 2\sqrt{\sum xy \cdot \frac{\sum xy}{9}} - \frac{2}{9} = \frac{2}{3} - \frac{2}{9} = \frac{4}{9}$$

$\Rightarrow P \ge \frac{4}{9} \Rightarrow P_{\min} = \frac{4}{9}$. ***Equality occurs if:*** $\begin{cases} x = y = z > 0 \\ x + y + z = 3 \end{cases} \Leftrightarrow x = y$

JP.189. Prove that:

$$\cot\frac{\pi}{26} - 4\sin\frac{5\pi}{13} = \sqrt{13 + 2\sqrt{13}}$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

We use the relationship:

$$\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13} = \frac{1}{2}\cot\frac{\pi}{26} \quad (1)$$

$$\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13} = \frac{1}{2}\sqrt{13 + 2\sqrt{13}} \quad (2)$$

The relationship from the problem statement is resulting by subtracting the relationships (1) and (2).

The relationship (1) result immediately from the known general relationship:

$$\sum_{k=1}^{n} \sin kx = \sin\frac{(n+1)x}{2}\sin\frac{nx}{2}\csc\frac{x}{2}$$

The relationship (2) results by summing relationships:

$$\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} = \frac{1}{2}\sqrt{\frac{13+3\sqrt{13}}{2}} \quad (3)$$

$$\sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13} = \frac{1}{2}\sqrt{\frac{13-3\sqrt{13}}{2}} \quad (4)$$

We demonstrate relationships (3) and (4).

We make the notation: $E = \sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13}$. *We obtain:*

$$E^2 = \frac{3}{2} + \frac{1}{2}\left(-\cos\frac{2\pi}{13} - \cos\frac{6\pi}{13} + \cos\frac{5\pi}{13}\right) + \cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13}$$

We make the notation: $F = \sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}$. *We obtain:*

$$F^2 = \frac{3}{2} - \frac{1}{2}\left(\cos\frac{4\pi}{13} - \cos\frac{3\pi}{13} - \cos\frac{\pi}{13}\right) - \cos\frac{3\pi}{13} - \cos\frac{6\pi}{13} + \cos\frac{4\pi}{13} + \cos\frac{5\pi}{13} - \cos\frac{\pi}{13} - \cos\frac{2\pi}{13}$$

Further, we use the following relationships: $\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} = \frac{1+\sqrt{13}}{4}$

$\cos\frac{5\pi}{13} - \cos\frac{6\pi}{13} - \cos\frac{2\pi}{13} = \frac{1-\sqrt{13}}{4}$. *For demonstration, we make the notations:*

$x = \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13}$, $y = \cos\frac{5\pi}{13} - \cos\frac{6\pi}{13} - \cos\frac{2\pi}{13}$. *We obtain:*

$S = x + y = \frac{1}{2}; P = xy = -\frac{3}{4}$. *From the equation:* $z^2 - Sz + P = 0$, *we have:*

$x = \frac{1+\sqrt{13}}{4}; y = \frac{1-\sqrt{13}}{4}$. *Then, we get:* $E^2 = \frac{13+3\sqrt{13}}{8}; F^2 = \frac{13-3\sqrt{13}}{8}$

so, relationships (3) and (4) is proved. Thus, the problem is solved.

JP.190. In ΔABC the following relationship holds:

$$\frac{9r}{8R} \le \frac{m_a w_a}{(b+c)^2} + \frac{m_b w_b}{(c+a)^2} + \frac{m_c w_c}{(a+b)^2} \le \frac{9}{16}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

$$\frac{9r}{8R} \le \frac{m_a l_a}{(b+c)^2} + \frac{m_b l_b}{(c+a)^2} + \frac{m_c l_c}{(a+b)^2} \le \frac{9}{16}$$

LHS: $\frac{m_a l_a}{(b+c)^2} + \frac{m_b l_b}{(c+a)^2} + \frac{m_c l_c}{(a+b)^2} \ge \frac{9r}{8R}$ *it follows from:*

Using $m_a l_a \ge s(s-a)$ *it follows* $\sum\frac{m_a l_a}{(b+c)^2} \ge \sum\frac{s(s-a)}{(b+c)^2}$,

$$\sum\frac{s(s-a)}{(b+c)^2} = \frac{s^4 + s^2(20Rr + 18r^2) + r^3(4R+r)}{4(s^2+r^2+2Rr)^2}$$

It remains to prove that: $\frac{s^4+s^2(20Rr+18r^2)+r^3(4R+r)}{4(s^2+r^2+2Rr)^2} \ge \frac{9r}{8R} \Leftrightarrow$

$$s^2[s^2(2R-9r)+r(40R^2-18r^2)]\ge r^3(28R^2+34Rr+9r^2)$$

We distinguish the following cases:

Case 1). If $(2R-9r)\ge 0$***, using Gerretsen's inequality:*** $s^2\ge 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)[(16Rr-5r^2)(2R-9r)+r(40R^2-18r^2)]\ge r^3(28R^2+34Rr+9r^2)$$

$$\Leftrightarrow 288R^3-713R^2r+292Rr^2-36r^3\ge 0\Leftrightarrow (R-2r)(288R^2-137Rr+18r^2)\ge 0$$

obviously from Euler's inequality $R\ge 2r$.

Case 2). If $(2R-9r)<0$***, with the observation that:*** $[s^2(2R-9r)+r(40R^2-18r^2)]>0$ ***using Gerretsen's inequality:*** $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2$***, it remains to prove that:***

$$(16Rr-5r^2)[(4R^2+4Rr+3r^2)(2R-9r)+r(40R^2-18r^2)]\ge r^3(28R^2+34Rr+9r^2)$$

$$\Leftrightarrow 32R^4+38R^3r-142R^2r^2-151Rr^3+54r^4\ge 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(34R^3+102R^2r+62Rr^2-27r^3)\ge 0$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

RHS: $\frac{m_al_a}{(b+c)^2}+\frac{m_bl_b}{(c+a)^2}+\frac{m_cl_c}{(a+b)^2}\le\frac{9}{16}$ ***it follows from:***

Using $m_al_a\le s(s-a)+\frac{1}{8}(b-c)^2$ ***it follows*** $\sum\frac{m_al_a}{(b+c)^2}\le\sum\frac{s(s-a)}{(b+c)^2}+\frac{1}{8}\sum\frac{(b-c)^2}{(b+c)^2}$,

$$\sum\frac{s(s-a)}{(b+c)^2}=\frac{s^4+s^2(20Rr+18r^2)+r^3(4R+r)}{4(s^2+r^2+2Rr)^2},$$

$\sum\frac{(b-c)^2}{(b+c)^2}=\frac{2s^6+s^4(3r^2-24Rr)-s^2r^2(52R^2+4r^2)-r^3(4R+r)^3}{s^2(s^2+r^2+2Rr)}$***, wherefrom***

$$\sum\frac{m_al_a}{(b+c)^2}\le\frac{4s^6+s^4(16Rr+39r^2)+s^2r^2(2r^2+4Rr-52R^2)-r^3(4R+r)^3}{8s^2(s^2+r^2+2Rr)}$$

It remains to prove that:

$$\frac{4s^6+s^4(16Rr+39r^2)+s^2r^2(2r^2+4Rr-52R^2)-r^3(4R+r)^3}{8s^2(s^2+r^2+2Rr)}\le\frac{9}{16}\Leftrightarrow$$

$$s^2[s^2(s^2+4Rr-60r^2)+r^2(140R^2+28Rr+5r^2)]+2r^3(4R+r)^3\ge 0$$

We distinguish the following cases:

Cases 1). If $(s^2+4Rr-60r^2)\ge 0$***, the inequality is obvious.***

Case 2). If** $(s^2+4Rr-60r^2)<0$, **with the observation that

$[s^2(s^2+4Rr-60r^2)+r^2(140R^2+28Rr+5r^2)]<0$, ***using the Blundon-***

Gerretsen's inequality $16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$

it remains to prove that:

$$\frac{R(4R+r)^2}{2(2R-r)}[(4R^2+4Rr+3r^2)(16Rr-5r^2+4Rr-60r^2)+r^2(140R^2+28Rr+5r^2)]+2r^3(4R+r)^3\ge 0$$

$$\Leftrightarrow 40R^4-20R^3r-70R^2r^2-99Rr^3-2r^4\ge 0$$

$\Leftrightarrow (R-2r)(40R^3+60R^2r+50Rr^2+r^3)\ge 0$, ***obviously from Euler's inequality*** $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{m_a}{w_a}\le\frac{b^2+c^2}{2bc}\Rightarrow\frac{m_aw_a}{(b+c)^2}\le\frac{(b^2+c^2)w_a^2}{2bc(b+c)^2}=\frac{2bc(b^2+c^2)}{(b+c)^4}\cdot\cos^2\frac{A}{2}$$

$\le\frac{1}{4}\cos^2\frac{A}{2}$; ***(because*** $\frac{2bc(b^2+c^2)}{(b+c)^4}\le\frac{1}{4}\Leftrightarrow(b+c)^4\ge 8bc(b^2+c^2)\Leftrightarrow(b-c)^4\ge 0$ ***(true))***

$$\Rightarrow\sum\frac{m_aw_a}{(b+c)^2}\le\frac{1}{4}\sum\cos^2\frac{A}{2}=\frac{1}{4}\sum\frac{1+\cos A}{2}\overset{\left(\sum\cos A\le\frac{3}{2}\right)}{\le}\frac{1}{4}\left(\frac{3}{2}+\frac{\frac{3}{2}}{2}\right)=\frac{1}{4}\left(\frac{3}{2}+\frac{3}{4}\right)=\frac{9}{16}$$

$$m_a\ge\frac{b^2+c^2}{4R}\ \textit{(etc)}\Rightarrow\sum\frac{m_aw_a}{(b+c)^2}\ge\frac{1}{4R}\sum\left\{\frac{b^2+c^2}{(b+c)^2}\cdot w_a\right\}\ge\frac{1}{4R}\sum\left(\frac{1}{2}\cdot w_a\right)=\frac{1}{8R}\sum w_a\overset{(*)}{\ge}\frac{9r}{8R}$$

$$(*)\Leftrightarrow\sum w_a\ge 9r$$

$$\because\sum w_a\ge\sum h_a=\frac{s^2+r^2+4Rr}{2R}\ge 9r\Leftrightarrow s^2+r^2+4Rr\ge 18Rr\Leftrightarrow s^2\ge 14Rr-r^2$$

It is true because:

$$s^2\ge 16Rr-5r^2\ge 14Rr-r^2\leftrightarrow 16Rr-5r^2\ge 14Rr-r^2\leftrightarrow 2Rr\ge 4r^2$$

$\leftrightarrow R\ge 2r$ ***(Euler) . Hence, (*) true. Proved.***

Solution 3 by Soumava Chakraborty-Kolkata-India

$$m_aw_a\ge\frac{b+c}{2}\cos\frac{A}{2}\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)=bc\cdot\frac{s(s-a)}{bc}\Rightarrow m_aw_a\overset{(1)}{\ge}s(s-a)$$

Similarly, $m_bw_b\overset{(2)}{\ge}s(s-b)$ ***&*** $m_cw_c\overset{(3)}{\ge}s(s-c)$

(1),(2),(3)$\Rightarrow\sum\frac{m_aw_a}{(b+c)^2}\ge\sum\frac{s(s-a)}{16R^2\cos^2\frac{A}{2}\cos^2\frac{B-C}{2}}\ge\sum\frac{bc\cos^2\frac{A}{2}}{16R^2\cos^2\frac{A}{2}}\left(\because 0<\cos^2\frac{B-C}{2}\le 1, etc\right)$

$$=\frac{\sum ab}{16R^2}\overset{?}{\geq}\frac{9r}{8R}\Leftrightarrow s^2+4Rr+r^2\overset{?}{\geq}18Rr$$

$$\Leftrightarrow s^2-14Rr+r^3\overset{?}{\geq}0\Leftrightarrow(s^2-16Rr+5r^2)+2r(R-2r)\overset{?}{\geq}0$$

$$\rightarrow \textbf{true} \because s^2-16Rr+5r^2\overset{Gerretsen}{\geq}0\ \&\ R-2r\overset{Euler}{\geq}0$$

$$\therefore \sum\frac{m_aw_a}{(b+c)^2}\geq\frac{9r}{8R}.\ \textbf{Again, }\sum\frac{m_aw_a}{(b+c)^2}\overset{G\leq A}{\leq}\frac{1}{4}\sum\frac{(m_a+w_a)^2}{(b+c)^2}$$

$$\overset{Tsintsifas}{\leq}\frac{1}{4}\sum\frac{\left(\frac{b^2+c^2}{2bc}w_a+w_a\right)^2}{(b+c)^2}=\frac{1}{4}\sum\frac{(b+c)^4w_a^2}{4b^2c^2(b+c)^2}$$

$$=\frac{1}{16}\sum(b+c)^2\left(\frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2}\right)=\frac{1}{8}\sum(1+\cos A)=\frac{1}{8}\left(3+1+\frac{r}{R}\right)=\frac{4R+r}{8R}\overset{?}{\leq}\frac{9}{16}$$

$$\Leftrightarrow 9R\overset{?}{\geq}8R+2r\Leftrightarrow R\overset{?}{\geq}2r\rightarrow \textbf{true (Euler)}\therefore\sum\frac{m_aw_a}{(b+c)^2}\leq\frac{9}{16}.\ \textbf{(proved)}$$

JP.191. Solve for real numbers:

$$\begin{cases}x^3+y^3=\sqrt{xy}(x^2+y^2)\\ 6\sqrt[3]{2x^2-2y+1}+4\cdot\sqrt[4]{3x^2\cdot y-2x^4}=2y^5-5\sqrt{xy}+13\end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Serban George Florin – Romania

$$\sqrt{xy}=\frac{x^3+y^3}{x^2+y^2}\underset{Mg\leq Ma}{\leq}\frac{x+y}{2}\Rightarrow 2(x^3+y^3)\leq(x+y)(x^2+y^2)$$

$$2x^3+2y^3\leq x^3+y^3+xy(x+y),x^3+y^3\leq xy(x+y)$$

$$(x+y)(x^2-xy+y^2)-xy(x+y)\leq 0,(x+y)(x-y)^2\leq 0$$

$$(x+y),(x-y)^2\geq 0.\ \textbf{If } x+y\leq 0\ \textbf{and } xy\geq 0\Rightarrow$$

$$\Rightarrow x,y\leq 0\Rightarrow\sqrt{xy}=\frac{x^3+y^3}{x^2+y^2}\leq 0\ \textbf{false, } x^3,y^3\leq 0$$

$$\Rightarrow x,y\geq\Rightarrow(x+y)(x-y)^2\geq 0\Rightarrow(x+y)(x-y)^2=0$$

$$\textbf{I. If } x+y=0,x,y\geq 0\Rightarrow x=y=0;6\cdot 1+4\cdot 0=0-0+13\ \textbf{false.}$$

$$\textbf{II. If } x=y\Rightarrow 6\sqrt[3]{2x^2-2x+1}+4\sqrt[4]{3x^3-2x^4}=2x^5-5x+13$$

$$2x^2-2x+1\geq 0,\Delta=-4<0,3x^3-2x^4\geq 0,x^3(3-2x)\geq 0$$

$$\Rightarrow 3-2x\geq 0\Rightarrow x\leq \frac{3}{2}\Rightarrow x\in\left[0,\frac{3}{2}\right]$$

$$\sqrt[3]{2x^2-2x+1}=\sqrt[3]{(2x^2-2x+1)\cdot 1\cdot 1}\underset{Mg\leq Ma}{\leq}\frac{2x^2-2x+1+1+1}{3}\Rightarrow 6\sqrt[3]{2x^2-2x+1}\leq 4x^2-4x+6$$

$$\sqrt[4]{3x^3-2x^4}=\sqrt[4]{(3x^3-2x^4)\cdot 1\cdot 1\cdot 1}\underset{Mg\leq Ma}{\leq}\frac{3x^3-2x^4+1+1+1}{4}\Rightarrow 4\sqrt[4]{3x^3-2x^4}\leq 3x^3-2x^4+3$$

$$\Rightarrow 2x^5-5x+13\leq 4x^2-4x+6+3x^3-2x^4+3$$

$$\Rightarrow 2x^5+2x^4-3x^3-4x^2-x+4\leq 0$$

$$(x^2-2x+1)(2x^3+6x^2+7x+4)\leq 0$$

$$2x^3+6x^2+7x+4>0,(\forall)x\in\left[0,\frac{3}{2}\right]$$

$$\Rightarrow (x^2-2x+1)\leq 0\Rightarrow (x-1)^2\leq 0$$

$$(x-1)^2\geq 0\Rightarrow (x-1)^2=0\Rightarrow x-1=0; x=1=y$$

$$\Rightarrow S=\{(1,1)\}$$

Solution 2 by Minh Tam Le-Vietnam

$$\textit{Let } \begin{matrix}\sqrt{x}=a\\ \sqrt{y}=b\end{matrix}\ (a,b\geq 0).\ \left.\begin{matrix}\textit{We have } a^6+b^6=ab(a^4+b^4)\\ \textit{But}: \begin{cases}5a^6+b^6\overset{AM-GM}{\geq}6a^5b\\ 5b^6+a^6\overset{AM-GM}{\geq}6ab^5\end{cases}\Rightarrow a^6+b^6\geq ab(a^4+b^4)\end{matrix}\right\}\Rightarrow$$

$$\Rightarrow a=b \text{ or } x=y.$$

$$\textit{If } x=y, 6\sqrt[6]{2x^2-2y+1}+4\sqrt[4]{3x^2y-2x^4}=2y^5-5\sqrt{xy}+13$$

$$\Leftrightarrow 6\sqrt[3]{2x^2-2x+1}+4\sqrt[4]{3x^3-2x^4}=2x^5-5x+13$$

$$LHS=2\cdot 3\sqrt[3]{2x^2-2x+1}+4\sqrt[4]{x^2\cdot x(3-2x)}\overset{AM-GM}{\leq}$$

$$\leq 2(2x^2-2x+1+1+1)+x^2+x+1+3-2x=5x^2-5x+10$$

$$RHS=x^5+x^5+1+1+1-5x+10\overset{AM-GM}{\geq}5x^2-5x+10$$

So, the equality holds if $x=1\Rightarrow y=1$. *Hence,* $x=1\wedge y=1$

JP.192. If $x,y,z>1$ then:

$$\log_x\left(\frac{y^5+z^5}{y^3+z^3}\right)+\log_y\left(\frac{z^5+x^5}{z^3+x^3}\right)+\log_z\left(\frac{x^5+y^5}{x^3+y^3}\right)\geq 6$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$x^5+y^5\ge xy(x^3+y^3)\Rightarrow \frac{x^5+y^5}{x^3+y^3}\ge xy;\ etc$$

$$\Rightarrow LHS\ge \log_z(xy)+\log_x(yz)+\log_y(xz)$$

$$=(\log_z x+\log_x z)+(\log_x y+\log_y x)+(\log_y z+\log_z y)$$

$$\overset{AM-GM}{\ge} 2\sqrt{\log_x z\cdot\log_z x}+2\sqrt{\log_x y\cdot\log_y x}+2\sqrt{\log_y z\cdot\log_z y}=2+2+2=6$$

Proved.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z>1$, we have: $\log_x\frac{y^5+z^5}{y^3+z^3}+\log_y\frac{z^5+x^5}{z^3+x^3}+\log_z\frac{x^5+y^5}{y^3+z^3}$

$$\text{If } \log_x\frac{(y^3+z^3)(y^2+x^2)}{2(y^3+x^3)}+\log_y\frac{(z^3+x^3)(z^2+x^2)}{2(z^3+x^3)}+\log_z\frac{(x^3+y^3)(x^2+y^2)}{2(x^3+y^3)}\ge 6$$

$$\text{If } \log_x\frac{y^2+z^2}{2}+\log_y\frac{z^2+x^2}{2}+\log_z\frac{x^2+y^2}{2}\ge 6$$

$$\text{If } \log_x yz+\log_y zx+\log_z xy\ge 6$$

$$\text{If } 3\sqrt[3]{\left(\frac{\log yz}{\log x}\right)\left(\frac{\log zx}{\log y}\right)\left(\frac{\log xy}{\log z}\right)}\ge 6$$

If $(\log y+\log z)(\log z+\log x)(\log x+\log y)\ge 8\log x\log y\log z$ and it's true.

Because $(\log y+\log z)(\log z+\log x)(\log x+\log y)$

$\ge 8\sqrt{(\log x\log y\log z)^2}=8\log x\log y\log z$. Therefore, it's true.

JP.193. In ΔABC the following relationship holds:

$$\frac{\tan^n\frac{A}{2}+\tan^n\frac{B}{2}}{\tan^{n+2}\frac{A}{2}+\tan^{n+2}\frac{B}{2}}+\frac{\tan^n\frac{B}{2}+\tan^n\frac{C}{2}}{\tan^{n+2}\frac{B}{2}+\tan^{n+2}\frac{C}{2}}+\frac{\tan^n\frac{C}{2}+\tan^n\frac{A}{2}}{\tan^{n+2}\frac{C}{2}+\tan^{n+2}\frac{A}{2}}\le 1+\frac{4R}{r};n\in\mathbb{N};n\ge 1$$

Proposed by Marian Ursărescu – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\tan^{n+2}\frac{A}{2}+\tan^{n+2}\frac{B}{2}=\tan^n\frac{A}{2}\cdot\tan^2\frac{A}{2}+\tan^n\frac{B}{2}\cdot\tan^2\frac{B}{2}\quad (*)$$

Suppose: $A\ge B\ge C\Rightarrow \tan\frac{A}{2}\ge\tan\frac{B}{2}\ge\tan\frac{C}{2}\Rightarrow\tan^n\frac{A}{2}\ge\tan^n\frac{B}{2}\ge\tan^n\frac{C}{2}$

By Chebyshev's inequality: (*) $\geq \frac{\left(\tan^n\frac{A}{2}+\tan^n\frac{B}{2}\right)\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}\right)}{2}$

Let $x = \tan\frac{A}{2}; y = \tan\frac{B}{2}; z = \tan\frac{C}{2}$ $(x, y, z > 0)$

$\left(x + y + z = \frac{4R+r}{s}; xyz = \frac{r}{s}\right) \Rightarrow \frac{x^n+y^n}{x^{n+2}+y^{n+2}} \leq \frac{2(x^n+y^n)}{(x^n+y^n)(x^2+y^2)} = \frac{2}{x^2+y^2}$***; (etc)***

$$\Rightarrow LHS \leq 2\left(\frac{1}{x^2 + y^2} + \frac{1}{y^2 + z^2} + \frac{1}{z^2 + x^2}\right) \overset{(AM-GM)}{\leq}$$

$\frac{1}{xy1} + \frac{1}{yz} + \frac{1}{zx} = \frac{x+y+z}{xyz} = \frac{4R+r}{s} \cdot \frac{s}{r} = \frac{4R+r}{r} = 1 + \frac{4R}{r}$***. Proved.***

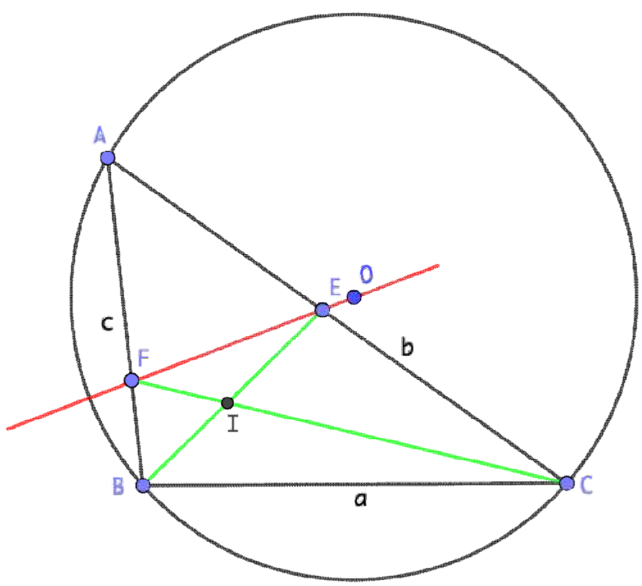
JP.194. In ΔABC; BE; CF internal bisectors;

$E \in (AC)$; $F \in (AB)$; O – circumcentre. Prove that:

E, O, F collinears $\Leftrightarrow \cos A = \cos B + \cos C$

Proposed by Marian Ursărescu – Romania

Solution by Thanasis Gakopoulos-Athens-Greece



E, O, F collinear $\leftrightarrow \cos A = \cos B + \cos C$; $\cos A = \frac{b^2+c^2-a^2}{2bc}$; $\cos B = \frac{a^2+c^2-b^2}{2ac}$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$\cos A = \cos B + \cos C \Leftrightarrow b^3 + c^3 - a^3 - a^2(b+c) + b^2(a-c) + c^2(a-b) = 0$ ***(1)***

PLAGIOGONAL system: $AB \equiv Ax, AC = Ay$

$$AF = f_1 = \frac{bc}{a+b}, AE = \frac{bc}{a+c} = e_2 \quad F(f_1, 0), E(0, e_2)$$

$$O(o_1, o_2) \quad o_1 = \frac{b^2c(a^2-b^2+c^2)}{16S^2}, o_2 = \frac{bc^2(a^2+b^2-c^2)}{16S^2}$$

E, O, F ***collinear*** $\leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ f_1 & o & o_1 \\ o & e_2 & o_2 \end{vmatrix} = 0 \leftrightarrow b^3 + c^3 - a^3 + b^2(a-c) + c^2(a-b) - a^2(b+c)$ ***(2)***

So, E, O, F ***collinear*** $\leftrightarrow$ ***(2) = (1)*** $\leftrightarrow \cos A = \cos B + \cos C$

JP.195. If $m \geq 0$ then in ΔABC the following relationship holds:

$$\frac{r_a \cdot r_b^{m+1}}{(r_b + r_c)^m} + \frac{r_b \cdot r_c^{m+1}}{(r_c + r_a)^m} + \frac{r_c \cdot r_a^{m+1}}{(r_a + r_b)^m} \geq \frac{s^2}{2^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai – Stanciu – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum \frac{r_a \cdot r_b^{m+1}}{(r_b + r_c)^m} = \sum \frac{(r_a r_b)^{m+1}}{(r_a r_b + r_a r_c)^m} \overset{Radon}{\geq} \frac{(\sum r_a r_b)^{m+1}}{2^m (\sum r_a r_b)^m}$$

$= \frac{\sum r_a r_b}{2^m} = \frac{s^2}{2^m}$. ***(Because:*** $\sum r_a r_b = s^2$***). Proved.***

SP.181. If $x, y, z > 0$ then:

$$\tan^{-1}\left(\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)}\right) + \tan^{-1}\left(\frac{y^4+z^4}{(y^2+z^2)(y^2-yz+z^2)}\right) +$$
$$+ \tan^{-1}\left(\frac{z^4+x^4}{(z^2+x^2)(z^2-zx+x^2)}\right) \geq \frac{3\pi}{4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have: $\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)} \geq 1 \ (x, y > 0)$

$$\Leftrightarrow x^4 + y^4 \geq (x^2+y^2)(x^2-xy+y^2) \Leftrightarrow x^4 + y^4 \geq x^4 - x^3y + 2x^2y^2 - xy^3 + y^4$$

$\Leftrightarrow xy(x-y)^2 \geq 0$ ***(∵ true:*** $x, y > 0$***)***

Similarly: $\frac{y^4+z^4}{(y^2+z^2)(y^2-yz+z^2)} \ge 1$ ***and:*** $\frac{x^4+z^4}{(x^2+z^2)(x^2-xz+z^2)} \ge 1$

$$\Rightarrow LHS \ge 3(\tan^{-1} 1) = 3 \cdot \frac{\pi}{4} \text{ (proved)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a, b \ge 0$. ***Now,*** $(a+b)(a^4+b^4) \ge (a^2+b^2)(a^3+b^3) \Leftrightarrow ab(a-b)^2(a+b) \ge 0$

Which is true. Hence $(a+b)(a^4+b^4) \ge (a^2+b^2)(a^3+b^3)$ ***is established***

$$\sum_{cyc} \tan^{-1}\left(\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)}\right) = \sum_{cyc} \tan^{-1}\left(\frac{(x+y)(x^4+y^4)}{(x^2+y^2)(x^3+y^3)}\right) \ge 3\tan^{-1}(1)$$

$$= \frac{3\pi}{4} \text{ (proved)}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b > 0$, ***we get:*** $a^4+b^4 = (a^2+b^2)(a^2+b^2) - 2a^2b^2$

$$= (a^2+b^2)\left(a^2+b^2-\frac{2a^2b^2}{a^2+b^2}\right) \ge (a^2+b^2)(a^2+b^2-ab) : -\frac{2a^2b^2}{a^2+b^2} \ge -ab$$

$$\Rightarrow \frac{a^4+b^4}{(a^2+b^2)(a^2-ab+b^2)} \ge 1, \forall a, b > 0$$

Hence, therefore $\arctan\left[\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)}\right] + \arctan\left[\frac{y^4+z^4}{(y^2+z^2)(y^2-yz+z^2)}\right] +$

$$+\arctan\left[\frac{z^4+x^4}{(z^2+x^2)(z^2-zx+x^2)}\right] \ge \arctan(1) + \arctan(1) + \arctan(1) =$$

$= \frac{3\pi}{4} : \frac{\arctan(1)=\pi}{4}$. ***Therefore, it is true.***

SP.182. If $f(x+\pi) = -f(x)$ and $f(-x) = f(x), f:(0,\infty) \to \mathbb{R}$, then:

$$\int_0^\infty f(x)\frac{\sin(x)}{x}dx = \int_0^{\frac{\pi}{2}} f(x)\cos(x)\,dx$$

Proposed by Shivam Sharma – New Delhi – India

Solution by Zaharia Burghelea – Romania

$$I=\int_0^\infty f(x)\frac{\sin x}{x}dx$$

$$f(-x)=f(x)\Rightarrow 2I=\int_{-\infty}^{\infty} f(x)\frac{\sin x}{x}dx=\sum_{n=-\infty}^{\infty}\int_{n\pi}^{(n+1)\pi} f(x)\frac{\sin x}{x}dx$$

$$f(x+\pi)=-f(x)\overset{x-n\pi=t}{\Rightarrow}\sum_{n=-\infty}^{\infty}\int_0^{\pi}(-1)^n f(t)\frac{(-1)^n\sin t}{t+n\pi}dt=\int_0^{\pi}\sum_{n=-\infty}^{\infty}\frac{\sin t}{t+n\pi}f(t)dt=$$

$$=\int_0^{\pi}\sum_{n=-\infty}^{\infty}\frac{\tan t}{t+n\pi}\cos t\cdot f(t)dt\overset{t=x}{=}\int_0^{\pi} f(x)\cos x\,dx$$

$$\Rightarrow I=\int_0^\infty f(x)\frac{\sin x}{x}dx=\int_0^{\frac{\pi}{2}} f(x)\cos x\,dx$$

Where the interchange of the sum and the integral is justified since the integrand is positive, also, we have:

$$\frac{\sin x}{x}=\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{x^2}{(k\pi)^2}\right)\Rightarrow \ln\left(\frac{\sin x}{x}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1-\frac{x^2}{(k\pi)^2}\right)$$

Differentiating with respect to x gives:

$$\cot x-\frac{1}{x}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2x}{x^2-(k\pi)^2}=\lim_{n\to\infty}\left(\frac{1}{x+\pi}+\frac{1}{x-\pi}+\cdots+\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right)$$

$$\Rightarrow \cot x=\sum_{n=-\infty}^{\infty}\frac{1}{x+n\pi}\Rightarrow\sum_{n=-\infty}^{\infty}\frac{\tan x}{x+n\pi}=1$$

SP.183. If $0<a<b$ then:

$$\frac{\int_a^b(\tan^{-1}x)\,dx}{\int_a^{\sqrt{ab}}(\tan^{-1}x)\,dx}>1+\sqrt{\frac{b}{a}}$$

Proposed by Daniel Sitaru – Romania

Solution by Avishek Mitra-West Bengal-India

$$\tan^{-1} x \geq -\frac{\pi}{2} \Rightarrow I_1 = \int_a^b \tan^{-1} x\, dx \geq \frac{\pi}{2}(a-b)$$

Similarly, $\Rightarrow \int_a^{\sqrt{ab}} \tan^{-1} x\, dx \geq \frac{\pi}{2}(a-\sqrt{ab})$

$$\Rightarrow \left(1+\sqrt{\frac{b}{a}}\right)\int_a^{\sqrt{ab}} \tan^{-1} x\, dx \geq \frac{\pi}{2}(a-\sqrt{ab})\left(1+\sqrt{\frac{b}{a}}\right) \Rightarrow \left(1+\sqrt{\frac{b}{a}}\right) I_2 \geq \frac{\pi}{2}(a-b)$$

Hence $I_1 - \left(1+\sqrt{\frac{b}{a}}\right) I_2 \geq 0 \Rightarrow \frac{I_1}{I_2} \geq 1+\sqrt{\frac{b}{a}} \Rightarrow \frac{I_1}{I_2} > \left(1+\sqrt{\frac{b}{a}}\right)$

SP.184. Let x, y, z be positive real numbers such that:

$x+y+z=3$. Prove that: $\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+12 \geq 5(xy+yz+zx)$.

Find the minimum value of:

$$T + \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}}{10}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

– ***Using AM-GM inequality we have:***

$$\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+x^3+x^3+x^3+1 \geq 10\cdot\sqrt[10]{\sqrt[6]{x}\cdot\sqrt[6]{x}\cdot\sqrt[6]{x}\cdot\sqrt[6]{x}\cdot\sqrt[6]{x}\cdot\sqrt[6]{x}\cdot x^3\cdot x^3\cdot x^3\cdot 1} = 10\cdot\sqrt[10]{x^{10}} = 10x$$

$$\Rightarrow 6\sqrt[6]{x}+3x^2+1 \geq 10x \Leftrightarrow 6\sqrt[6]{x} \geq 10x-3x^3-1$$

\+ ***Similar:*** $6\sqrt[6]{y} \geq 10y-3y^3-1; 6\sqrt[6]{z} \geq 10z-3z^3-1$

– ***Hence*** $\Rightarrow 6(\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}) \geq 10(x+y+z)-3(x^3+y^3+z^3)-3$

$\Leftrightarrow 6(\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}) \geq 10\cdot 3-3(x^3+y^3+z^3)-3 = 27-3(x^3+y^3+z^3)$ ***(because*** $x+y+z=3$***)***

$$\Leftrightarrow 6(\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}) \geq 27-3(x^3+y^3+z^3) \quad \textbf{(1)}$$

\+***Other, because*** $\begin{cases} x,y,z>0 \\ x+y+z=3 \end{cases} \Rightarrow 0 < x,y,z < 3 \Rightarrow x-3<0; y-3<0; z-3<0.$

Hence: $(x-3)(x-1)^2+(y-3)(y-1)^2+(z-3)(z-1)^2 \leq 0$

$$\Leftrightarrow (x-3)(x^2-2x+1)+(y-3)(y^2-2y+1)+(z-3)(z^2-2z+1)\le 0$$

$$\Leftrightarrow x^3-5x^2+7x-3+y^3-5y^2+7y-3+z^3-5z^2+7z-3\le 0$$

$$\Leftrightarrow x^3+y^3+z^3\le 5(x^2+y^2+z^2)-7(x+y+z)+9=5[(x+y+z)^2-2(xy+yz+zx)]-7(x+y+z)+9$$

$$\Leftrightarrow x^3+y^3+z^3\le 5(x+y+z)^2-10(xy+yz+zx)-7(x+y+z)+9$$

$$\Leftrightarrow x^3+y^3+z^3\le 5\cdot 3^2-10(xy+yz+zx)-7\cdot 3+9=33-10(xy+yz+zx)$$

$$\Leftrightarrow 27-3(x^3+y^3+z^3)\ge 27-3[33-10(xy+yz+zx)]=30(xy+yz+zx)-72\quad (2)$$

– *Let (1), (2):* $\Rightarrow 6(\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z})\ge 30(xy+yz+zx)-72\Rightarrow \sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}\ge 5(xy+yz+zx)-12$

$\Rightarrow \sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+12\ge 5(xy+yz+zx)$ ***and we get the result.***

– ***Using Cauchy Schwarz inequality, we have:***

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{x^2}{x(y+z)}+\frac{y^2}{y(z+x)}+\frac{z^2}{z(x+y)}\ge\frac{(x+y+z)^2}{x(y+z)+y(z+x)+z(x+y)}$$

$+$ ***Hence*** $T=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}}{10}\ge\frac{(x+y+z)^2}{2(xy+yz+zx)}+\frac{\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}}{10}$

– ***Using AM-GM inequality and inequality:*** $\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}\ge 5(xy+yz+zx)-12$

$$\Rightarrow T\ge\frac{9}{2(xy+yz+zx)}+\frac{5(xy+yz+zx)-12}{10}=\frac{1}{2}\cdot\left(\frac{9}{xy+yz+zx}+xy+yz+zx\right)-\frac{6}{5}\ge$$

$$\ge\frac{1}{2}\cdot 2\cdot\sqrt{\frac{9}{xy+yz+zx}\cdot(xy+yz+zx)}-\frac{6}{5}=\frac{1}{2}\cdot 2\cdot 3-\frac{6}{5}=3-\frac{6}{5}=\frac{9}{5}$$

$\Rightarrow T\ge\frac{9}{5}\Rightarrow T_{\min}=\frac{9}{5}$. ***Equality occurs if*** $\begin{cases}x+y+z=3\\ x=y=z>0\\ xy+yz+zx=3\end{cases}\Leftrightarrow x=y=z=1$

Hence, minimum value of $T=\frac{9}{5}$ ***when*** $x=y=z=1$.

SP.185. Let a,b,c be positive real numbers such that:

$12a+8b+6c=3abc$. **Find the minimum value of:**

$$T=\frac{a^3+20}{a}+\frac{b^4+249}{b}+\frac{c^4+272}{c^2}.$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by proposer

- By AM-GM inequality we have:

$$T=\frac{a^3+20}{a}+\frac{b^4+249}{b}+\frac{c^4+272}{c^2}=a^2+\frac{20}{a}+b^3+\frac{249}{b}+c^2+\frac{272}{c^2}$$

$$T=\left(a^2+\frac{8}{a}+\frac{8}{a}\right)+\left(b^3+\frac{81}{b}+\frac{81}{b}+\frac{81}{b}\right)+\left(c^2+\frac{256}{c^2}\right)+\frac{4}{a}+\frac{6}{b}+\frac{16}{c^2}\geq$$

$$\geq 3\cdot\sqrt[3]{a^2\cdot\frac{8}{a}\cdot\frac{8}{a}}+4\sqrt[4]{b^3\cdot\frac{81}{b}\cdot\frac{81}{b}\cdot\frac{81}{b}}+2\sqrt{c^2\cdot\frac{256}{c^2}}+\frac{4}{a}+\frac{6}{b}+\left(\frac{16}{c^2}+1\right)-1$$

$$=3\sqrt[3]{64}+4\sqrt[4]{81^3}+2\sqrt{256}+\frac{4}{a}+\frac{6}{b}+2\sqrt{\frac{16}{c^2}}-1=\frac{4}{a}+\frac{6}{b}+\frac{8}{c}+151$$

$$\Rightarrow T\geq 2\left(\frac{2}{a}+\frac{3}{b}+\frac{4}{c}\right)+151\geq 2\sqrt{3\left(\frac{2}{a}\cdot\frac{3}{b}+\frac{3}{b}\cdot\frac{4}{c}+\frac{4}{c}\cdot\frac{2}{a}\right)}+151=$$

$$=2\sqrt{3\left(\frac{6}{ab}+\frac{12}{bc}+\frac{8}{ca}\right)}+151$$

$$\Rightarrow T\geq 2\sqrt{3\cdot\frac{12a+8b+6c}{abc}}+151=2\sqrt{3\cdot\frac{3abc}{abc}}+151=157\Rightarrow T\geq 157\Rightarrow T_{\min}=157$$

+ ***Equality occurs if*** $\begin{cases}a,b,c>0;12a+8b+6c=3abc\\ a^2=\frac{8}{a};b^3=\frac{81}{b};c^2=\frac{256}{c^2}\\ \frac{2}{a}=\frac{3}{b}=\frac{4}{c}\end{cases}\Leftrightarrow\begin{cases}a=2\\ b=3\\ c=4\end{cases}$

Hence, the minimum value of T ***is*** 157 ***then*** $a=2;b=3;c=4$.

Solution 2 by Michael Sterghiou-Greece

$$T=\frac{a^3+20}{a}+\frac{b^4+249}{b}+\frac{c^4+272}{c^2}\quad (1)$$

$$\frac{a^3+20}{a}=a^2+\frac{20}{a}=a^2+\frac{8}{a}+\frac{8}{a}+\frac{4}{a}\overset{AM-GM}{\geq}3\sqrt[3]{64}+\frac{4}{a}=12+\frac{4}{a}\quad (2)$$

$$\frac{b^4+249}{b}=b^3+\frac{249}{b}=b^3+\frac{81}{b}+\frac{168}{b}\overset{AM-GM}{\geq}18b+\frac{162}{b}+\frac{6}{b}\overset{AM-GM}{\geq}108+\frac{6}{b}\quad (3)$$

$$\frac{c^4+272}{c^2}=c^2+\frac{272}{c^2}=c^2+\frac{256}{c^2}+\frac{16}{c^2}\overset{AM-GM}{\geq}32+\frac{16}{c^2}\quad (3)$$

(1)+(2)+(3)$\rightarrow T\geq 152+\frac{4}{a}+\frac{6}{b}+\frac{16}{c^2}$ ***(4) Let*** $x=12a,y=8b,z=6c$

(c)$\rightarrow x+y+z=\frac{xyz}{192}$ ***and (4)***$\rightarrow T\geq 152+\frac{48}{x}+\frac{48}{y}+\frac{576}{z^2}=$

$= 152 + 48 \cdot \left(\frac{1}{x} + \frac{1}{y} + \frac{12}{z^2}\right)$ ***(5). We will find the minimum of***

$f(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{12}{z^2}$ ***under the constraint*** $x + y + z = \frac{xyz}{192}\ (c')$

by using the Lagrange multiplier method. Consider the function

$$L_0(x, y, z, \lambda) = \frac{1}{x} + \frac{1}{y} + \frac{12}{z^2} + \lambda\left(x + y + z - \frac{xyz}{192}\right)$$

$\frac{\partial L_0}{\partial x} = -\frac{1}{x^2} - \frac{\lambda}{192}yz + \lambda = 0$ ***(6),*** $\frac{\vartheta L_0}{\vartheta y} = -\frac{1}{y^2} - \frac{\lambda}{192}xz + \lambda \overset{(7)}{=}, \frac{\vartheta L_0}{\vartheta z} = -\frac{24}{z^3} - \frac{\lambda xy}{192} + \lambda \overset{(8)}{=} 0$

and $\frac{\vartheta L_0}{\vartheta \lambda} = x + y + z - \frac{xyz}{192} = 0$ ***(9) From (6) and (7)*** $\rightarrow x\left(x - \frac{xyz}{192}\right) = \frac{1}{\lambda}$ ***and***

$y\left(y - \frac{xyz}{192}\right) = \frac{1}{\lambda}$ ***hence*** $x(y + z) = -\frac{1}{\lambda}$ ***and*** $y(x + z) = -\frac{1}{\lambda} \rightarrow x = y\ \ (z > 0)$

$(c') \rightarrow 2x + z = \frac{x^2 z}{192} \rightarrow zx^2 - 384x - 192z = 0$ ***(10) and from either (6), (7)***

and (8) with $x = y$ ***we get*** $zx^2 - 192x - 16z^2 = 0$ ***(11). From (11)–(10)→***

$192x + 192z - 16z^2 = 0 \rightarrow x = z\left(\frac{z}{12} - 1\right)$***. As*** $x.0 \rightarrow z > 12$***. Now (10) becomes***

$(z - 2y)(z^2 + 12z - 96) = 0$ ***giving*** $z = 2y$ ***and two roots*** < 12 ***and not acceptable.***

With $z = 2y$ ***we have*** $x = y = 2y$ ***and*** $\lambda = -\frac{1}{2 \cdot 2y^2}$

The point $(x_0, y_0, z_0) = (24, 24, 24)$ ***is global constrained min for*** $f(x, y, z)$

as $d^2 L_0 > 0$ ***(All 2nd derivatives*** > 0 ***and mixed 2nd order derivatives*** > 0***)***

$f(x_0, y_0, z_0) = f(24, 24, 24) = \frac{1}{24} + \frac{1}{24} + \frac{12}{2y^2} = \frac{5}{48}$ ***hence*** $T \geq 152 + 48 \cdot \frac{5}{48} = 157$

$$T_{\min} = 157$$

SP.186. Let m_a, m_b and m_c be the lengths of the medians of an acute triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{8\sqrt{3}r}{3R^3} \leq \frac{\cot\frac{A}{2}}{m_a^2} + \frac{\cot\frac{B}{2}}{m_b^2} + \frac{\cot\frac{C}{2}}{m_c^2} \leq \frac{\sqrt{3}R}{6 \cdot r^3}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Marian Ursărescu-Romania

In any acute triangle we have $m_a \le 2R\cos^2\frac{A}{2}$ *(1)*

$$\frac{\cot\frac{A}{2}}{m_a^2}+\frac{\cot\frac{B}{2}}{m_b^2}+\frac{\cot\frac{C}{2}}{m_c^2}\ge 3\sqrt[3]{\frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}{m_a^2m_b^2m_c^2}}\quad (2)$$

From (1)+ (2)$\Rightarrow \sum\frac{\cot\frac{A}{2}}{m_a^2}\ge 3\sqrt[3]{\frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}{64R^6\cos^4\frac{A}{2}\cos^4\frac{B}{2}\cdot\cos^4\frac{C}{2}}}$ *(3)*

From (3) we must show: $3\sqrt[3]{\frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}{64R^6\cdot\cos^4\frac{A}{2}\cos^4\frac{B}{2}\cos^4\frac{C}{2}}}\ge\frac{8\sqrt{3}r}{3R^3}\Leftrightarrow\frac{\sqrt{3}}{4R^2}\sqrt[3]{\frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}{\cos^4\frac{A}{2}\cos^4\frac{B}{2}\cos^4\frac{C}{2}}}\ge\frac{8r}{3R^3}\Leftrightarrow$

$$\sqrt{3}\sqrt[3]{\frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}{\cos^4\frac{A}{2}\cos^4\frac{B}{2}\cos^4\frac{C}{2}}}\ge\frac{2^5r}{27R}\quad (4)$$

But $\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}=\frac{s}{r}$ *(5)*

and $\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}=\frac{s^2}{16R^2}$ *(6)*

From (4)+(5)+(6) $\Rightarrow 3\sqrt{3}\frac{\frac{s}{r}}{\frac{s^4}{2^8R^4}}\ge\frac{2^{15}r^3}{27R^3}\Leftrightarrow$

$3\sqrt{3}\frac{2^8R^4}{s^3r}\ge 2^{15}\cdot\frac{r^3}{27R^3}\Leftrightarrow 3^4\sqrt{3}R^7\ge 2^7\cdot s^3r^4$ *(7)*

But $R\ge 2r\Rightarrow R^4\ge 2^4r^4$ *(8) and* $R\ge\frac{2}{3\sqrt{3}}s\Rightarrow 3\sqrt{3}R\ge 2s\Rightarrow 3^4\sqrt{3}R^3\ge 2^3s$ *(9)*

From (8)+(9)⇒ (7) it is true. Now, we have: $m_a\ge\frac{b+c}{2}\cdot\cos\frac{A}{2}\ge\sqrt{bc}\cos\frac{A}{2}\Rightarrow$

$$m_a^2\ge bc\cos\frac{A}{2}\Rightarrow\frac{1}{m_a^2}\le\frac{1}{bc\cos\frac{A}{2}}\Rightarrow\frac{\cot\frac{A}{2}}{m_a^2}\le\frac{1}{bc\cdot\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\frac{2}{bc\cdot\sin A}=\frac{4R}{abc}\Rightarrow$$

$\sum\frac{\cot\frac{A}{2}}{m_a^2}\le\frac{12R}{abc}\Rightarrow$ *we must show:* $\frac{12R}{abc}\le\frac{\sqrt{3}R}{6r^3}\Leftrightarrow abc\ge\frac{2^3\cdot 3^2\cdot r^3}{\sqrt{3}}$*. But* $abc=4sRr\Rightarrow$

$sRr\ge\frac{2\cdot 3^2}{\sqrt{3}}r^3\Leftrightarrow sR\ge 2\cdot 3\sqrt{3}r^2$ *true, because* $R\ge 2r$ *and* $s\ge 3\sqrt{3}r$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{8\sqrt{3}r}{3R^3}\overset{(1)}{\le}\sum\frac{\cot\frac{A}{2}}{m_a^2}\overset{(2)}{\le}\frac{\sqrt{3}R}{6\cdot r^3}$$

$$\sum\frac{\cot\frac{A}{2}}{m_a^2}=\sum\frac{s}{r_a m_a^2}=\sum\frac{s(s-a)}{rsm_a^2}$$

$$\overset{m_a^2\ge s(s-a),etc}{\le}\sum\frac{s(s-a)}{rs^2(s-a)}=\frac{3}{rs}\overset{?}{\le}\frac{\sqrt{3}R}{6r^3}\Leftrightarrow\sqrt{3}Rs\underset{(2i)}{\overset{?}{\ge}}18r^2$$

But $R\overset{Euler}{\ge}2r$ **&** $s\overset{Mitrinovic}{\ge}3\sqrt{3}r\Rightarrow\sqrt{3}Rs\ge\sqrt{3}(2r)(3\sqrt{3}r)=18r^2$

$\Rightarrow$ ***(2i)*** $\Rightarrow$ ***(2) is true. Again,*** $\sum\frac{\cot\frac{A}{2}}{m_a^2}=\sum\frac{s}{r_a m_a^2}\underset{(1i)}{\overset{Bergstrom}{\ge}}\frac{9s}{\sum r_a m_a^2}$

WLOG, we may assume $a\ge b\ge c\therefore r_a\ge r_b\ge r_c$ **&** $m_a^2\le m_b^2\le m_c^2$

$\therefore$ ***Chebyshev & (1i)*** $\Rightarrow\sum\frac{\cot\frac{A}{2}}{m_a^2}\ge\frac{9s}{\frac{1}{3}(\sum r_a)(\sum m_a^2)}=\frac{9s}{\frac{1}{3}(4R+r)\frac{3}{4}\sum a^2}$

$$=\frac{36s}{(4R+r)(\sum a^2)}\overset{Leibnitz}{\ge}\frac{36s}{(4R+r)(9R^2)}$$

$\overset{Euler}{\ge}\frac{4s}{\frac{9R^3}{2}}=\frac{8s}{9R^3}\overset{Mitrinovic}{\ge}\frac{8\cdot3\sqrt{3}r}{9R^3}=\frac{8\sqrt{3}r}{3R^3}\Rightarrow$ ***(1) is true (Proved)***

SP.187. Let a,b,c be the lengths of sides in a triangle such that $a^2+b^2+c^2=3$. Find the maximum value of:

$$P=\frac{1}{3a+bc}+\frac{1}{3b+ca}+\frac{1}{3c+ab}+ab+bc+ca$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by proposer

By AM-GM inequality we have:

$$a^2+b^2+c^2+3=(a^2+1)+(b^2+1)+(c^2+1)\ge2a+2b+2c=2(a+b+c)$$

$$\Rightarrow6\ge2(a+b+c)\Rightarrow3\ge a+b+c\quad(a^2+b^2+c^2=3)$$

$$+\textit{Hence}\begin{cases}3a+bc\ge(a+b+c)a+bc=a(a+b)+c(a+b)=(a+b)(a+c)\\3b+ca\ge(a+b+c)b+ca=b(b+c)+a(b+c)=(b+c)(b+a)\\3c+ab\ge(a+b+c)c+ab=c(c+a)+b(c+a)=(c+a)(c+b)\end{cases}$$

$$\Rightarrow\frac{1}{3a+bc}+\frac{1}{3b+ca}+\frac{1}{3c+ab}\le\frac{1}{(a+b)(a+c)}+\frac{1}{(b+c)(b+a)}+\frac{1}{(c+a)(c+b)}=$$

$$= \frac{2(a+b+c)}{(a+b)(b+c)(c+a)}$$

$$\Rightarrow P = \frac{1}{3a+bc} + \frac{1}{3b+ca} + \frac{1}{3c+ab} + ab + bc + ca \le \frac{2(a+b+c)}{(a+b)(b+c)(c+a)} + ab + bc + ca \quad (1)$$

– *We have* $\frac{(a+b)(b+c)(c+a)}{8} \ge \frac{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}}{8} = \frac{8abc}{8} = abc$

$$\Rightarrow (a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a) + abc \le (a+b)(b+c)(c+a) + \frac{(a+b)(b+c)(c+a)}{8}$$

$$\Rightarrow (a+b+c)(ab+bc+ca) \le \frac{9(a+b)(b+c)(c+a)}{8} \Leftrightarrow (a+b)(b+c)(c+a) \ge \frac{8(a+b+c)(ab+bc+ca)}{9}$$

$$\Leftrightarrow \frac{2(a+b+c)}{(a+b)(b+c)(c+a)} \le \frac{2(a+b+c)}{\frac{8(a+b+c)(ab+bc+ca)}{9}} = \frac{9}{4(ab+bc+ca)} \quad (2)$$

– *Let (1), (2)* $\Rightarrow P \le \frac{9}{4(ab+bc+ca)} + ab + bc + ca = \frac{9}{4t} + t$ *(if* $t = ab + bc + ca > 0$*)* *(3)*

– *Other, by inequality in a triangle, we have* $\begin{cases} b + c > a \\ c + a > b \\ a + b > c \end{cases} \Rightarrow \begin{cases} a(b+c) > a^2 \\ b(c+a) > b^2 \\ c(a+b) > c^2 \end{cases}$

$$\Rightarrow a(b+c) + b(c+a) + c(a+b) > a^2 + b^2 + c^2 = 3 \Rightarrow 2(ab+bc+ca) > 3 \Leftrightarrow 2t > 3 \Leftrightarrow 2t - 3 > 0$$

– *Other, by inequality* $a^2 + b^2 + c^2 \ge ab + bc + ca \Rightarrow 3 \ge ab + bc + ca = t$

$\Rightarrow 3 \ge t \Rightarrow t - 3 \le 0$, *with* $2t - 3 > 0 \Rightarrow (t-3)(2t-3) \le 0 \Leftrightarrow 2t - 9t + 9 \le 0 \Leftrightarrow \frac{t}{2} + \frac{9}{4t} \le \frac{9}{4}$

$$\Rightarrow P \le \frac{9}{4t} + t \le \frac{9}{4} + \frac{t}{2} \le \frac{9}{4} + \frac{3}{2} = \frac{15}{4} \Rightarrow P \le \frac{15}{4} \Rightarrow P_{\max} = \frac{15}{4}$$

\+ *Equality occurs if* $\begin{cases} a + b + c = 3 \\ a = b = c > 0 \end{cases} \Leftrightarrow a = b = c = 1.$

Hence, maximum value of P *is:* $\frac{15}{4}$ *then* $a = b = c = 1.$

Solution 2 by Michael Sterghiou-Greece

$$P = \left(\sum_{cyc} \frac{1}{3a+bc}\right) + q \quad (1)$$

Let $(p,q,r) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)$. $\sum_{cyc} a^2 = p^2 - 2q = 3$. *First, we show that:*

$2q > \sum_{cyc} a^2$ *(2): As* $b < a + c$ *(triangle)*→ $a > b - c$ *and as* $c < a + b \to a > c - b$ *or*

$|a| > |b - c|$ *and* $a^2 > (b-c)^2$ *(3). Now (2) can be written as* $\sum_{cyc} a^2 \ge \sum_{cyc}(a-b)^2$

which holds true because of (3) and the cyclic relations.

(1)→ $\frac{\sum_{cyc}(3b+ca)(3c+ab)}{\prod_{cyc}(3a+bc)} + q \le \frac{15}{4}$ *as we will show.*

or $\frac{pr+3pq-9r+9a}{36r+9q^2-18pr+r^2}+q\le\frac{15}{4}$ *(4) [Note: we have replaced the terms* $(\sum a^2b)+(\sum a^2c)$ *with* $pq-3r$*]. (4) reduces to*

$-72pqr+12pq+274pr+36q^3-135q^2+4qr^2+144qr+36q-15r^2-576r\le 0$ *or*

$36q^3-135q^2+36q+(4q-15)r^2+(274p-72pq+144q-576)r+2pq\le 0.$

We have $p\le 3; q\le 3$ *(due to* $\sum_{cyc}x^3=3$*) and* $q\ge\frac{3}{2}$ *(from (2)) so* $(4q-15)<0$

We can show easily that $(274p-72pq+144q-576)\ge 0$ *(we can replace* p *by* $\sqrt{2q+3}$ *and use analysis with* $q\in\left[\frac{3}{2,3}\right]$*) and as* $r>\frac{1}{9}p(4q-p^2)$ *[from Schur 3rd degree,* $q\le\frac{p^3+9r}{4p}$*] and* $r<\left(\frac{q^2}{3p}\right)$ $(q^2\ge 3pr)$ *we get the stronger inequality:*

$$f(q)=12\sqrt{2q+3}\cdot q+36q^3-135q^4+36q+(4q-15)\cdot\frac{1}{81}(2q+3)(2q-3)^2+$$

$$+\left(274\sqrt{2q+3}+144q-72q\sqrt{2q+3}-576\right)\cdot\left(\frac{q^2}{3\sqrt{2q+3}}\right)\le 0\ \textit{(5)}$$

$$\textit{(5)}\rightarrow\frac{1}{81\sqrt{2q+3}}\begin{bmatrix}32\sqrt{2q+3}\cdot q^4+804\sqrt{2q+3}q^3+3888q^3-3429\sqrt{2q+3}\,\cdot\\ \cdot\, q^2-13608q^2+3294\sqrt{2q+3}\cdot q+2916q-405\sqrt{2q+3}\end{bmatrix}\le 0\ \textit{(6)}$$

Keeping the function in the brackets as $\frac{1}{81}\sqrt{2q+3}>0$ *and replacing* $p=\sqrt{2q+3}$ *we get:*

$$f(p)=2p^9+\frac{153}{2}p^7+489p^6-\frac{6615}{4}p^5-7776p^4+9288p^3+34922p^2-\frac{62451}{4}p-48114\le 0\ \textit{(7)}$$

with $p\in\left[\sqrt{\frac{9}{2}},3\right]$ *as* $p^2\ge 3q$*. (7) reduces to:*

$f(p)=\frac{1}{4}(p-3)g(p)$ *where* $g(p)=8p^8+24p^7+378p^6+3078p^5+2619p^4-$ $-23247p^3-32589p^2+42201p+64152$*. We can show that* $q(p)>0$:

$$g^{(4)}(p)=24\cdot(560p^4+840p^3+5670p^2+15390p+2619)>0$$

Hence $g^{(3)}(p)\uparrow\rightarrow g^{(3)}(p)>g\left(\sqrt{\frac{9}{2}}\right)>0\rightarrow g''(p)\uparrow$ *etc up to* $g(p)$

Therefore $f(p)\le 0$ *as* $p\le 3$ *and* $g(p)>0$*. We are done!*

SP.188. In $\Delta ABC, r_a, r_b, r_c$ are exradii. Prove that:

$$\frac{r_a}{r_b}\cdot\sin^2\frac{A}{2}+\frac{r_b}{r_c}\cdot\sin^2\frac{B}{2}+\frac{r_c}{r_a}\cdot\sin^2\frac{C}{2}\geq\frac{3}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\frac{r_a}{r_b}\cdot\sin^2\frac{A}{2}=\frac{s-b}{s-a}\cdot\frac{(s-b)(s-c)}{bc};\frac{r_b}{r_c}\cdot\sin^2\frac{B}{2}=\frac{s-c}{s-b}\cdot\frac{(s-a)(s-c)}{ac}$$

$\frac{r_c}{r_a}\cdot\sin^2\frac{C}{2}=\frac{s-a}{s-c}\cdot\frac{(s-a)(s-b)}{ab}$. ***Must show that:***

$$a(s-b)^3(s-c)^2+b(s-c)^3(s-a)^2+c(s-a)^3(s-b)^3$$

$$\geq\frac{3}{4}abc(s-a)(s-b)(s-c)\quad (*)$$

(Let $x=s-a;y=s-b;z=s-c\Rightarrow x+y+z=s\Rightarrow$

$a=y+z;b=x+z;c=x+y$***)***

(*) $\Leftrightarrow 4[(y+z)y^3z^2+(x+z)z^3x^2+(x+y)x^3y^2]\geq 3xyz(x+y)(y+z)(z+x)$

$$\Leftrightarrow 4[y^4z^2+y^3z^3+x^3z^3+x^2z^4+y^2x^4+x^3y^3]\geq$$

$$\geq 3\{2(xyz)^2+zy^2x^3+yx^2z^3+xz^2y^3+yz^2x^3+zx^2y^3+xy^2z^3\}\quad (**)$$

$$3(y^3z^3+x^3z^3+x^3y^3)\overset{(1)}{\geq}3(yx^2z^3+zy^2x^2+xz^2y^3)$$

$$(\because X^3+Y^3+Z^3\geq XY^2+YZ^2+ZX^2)$$

$$3(z^2y^4+y^2x^4+x^2z^4)=3\{(zy^2)^2+(yx^2)^2+(xz^2)^2\}$$

$$\overset{(2)}{\geq}3(zx^2y^3+yz^2x^3+xy^2z^3)$$

$$y^4z^2+x^2z^4+y^2x^4+x^3y^3+y^3z^3+z^3x^3\overset{(AM-GM)}{\geq}$$

$$6\sqrt[6]{(xyz)^{12}}=3\cdot 2(xyz)^2\quad (3)$$

From (1)+(2)+(3)⇒ (**) true ⇒ (*) true.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{r_a^2\sin^2\frac{A}{2}}{r_ar_b}\overset{Bergstrom}{\geq}\frac{\left(\sum r_a\sin\frac{A}{2}\right)^2}{s^2}$$

$$=\frac{\left(\sum s\frac{\sin^2\frac{A}{2}}{\cos\frac{A}{2}}\right)^2}{s^2}=\left(\sum\frac{\sin^2\frac{A}{2}}{\cos\frac{A}{2}}\right)^2\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\sum\frac{\sin^2\frac{A}{2}}{\cos\frac{A}{2}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum\frac{1-\cos^2\frac{A}{2}}{\cos\frac{A}{2}}\overset{?}{\geq}\frac{\sqrt{3}}{2}$$

$$\Leftrightarrow\sum\sec\frac{A}{2}-\sum\cos\frac{A}{2}\underset{(1)}{\overset{?}{\geq}}\frac{\sqrt{3}}{2}$$

Now, $\sum\sec\frac{A}{2}\underset{(a)}{\overset{Jensen}{\geq}}3\sec\frac{\pi}{6}=2\sqrt{3}$ *(*$\because f(x)=\sec\frac{x}{2}$ *is convex* $\forall x\in(0,\pi)$*)*

Also, $\sum\cos\frac{A}{2}\overset{Jensen}{\leq}3\cos\frac{\pi}{6}=3\frac{\sqrt{3}}{2}$ *(*$\because f(x)=\cos\frac{x}{2}$ *is convex* $\forall x\in(0,\pi)$*)*

$$\Rightarrow-\sum\cos\frac{A}{2}\overset{(b)}{\geq}-3\frac{\sqrt{3}}{2}$$

(a)+(b)⇒ LHS of (1) $\geq\left(2-\frac{3}{2}\right)\sqrt{3}=\frac{\sqrt{3}}{2}\Rightarrow$ ***(1) is true (Done)***

SP.189. Let be $f:\mathbb{R}\to\mathbb{R};f(x)=x^{2p+1}+e^{x-1};p\in\mathbb{N};p\neq 0;$

$a_n=f^{-1}\left(2+\frac{1}{n}\right);n\in\mathbb{N};n\neq 0$**. Find:**

$$\Omega=\lim_{n\to\infty}a_n^n$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer

$f'(x)=(2p+1)x^{2p}+e^{xy}>0,\forall x\in\mathbb{R}\Rightarrow f$ ***increasing*** $\Rightarrow f$ ***injective***

x	$-\infty$ $\qquad\qquad$ $+\infty$
$f'(x)$	+ + + + + + + + + + + + + + + + + +
$f(x)$	$-\infty$ ↗ $+\infty$

$Im\ f=\mathbb{R}\Rightarrow f$ ***surjective*** $\Rightarrow f$ ***bijective***

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}f^{-1}\left(2+\frac{1}{n}\right)=f^{-1}\left(\lim_{n\to\infty}2+\frac{1}{n}\right)=f^{-1}(2)$ ***(1)***

$f^{-1}(2)=x\Leftrightarrow f(x)=2\Leftrightarrow x^{2p+1}+e^{x-1}=2\Rightarrow x=1$ ***(2)***

(1)+(2)⇒ $\lim_{n\to\infty}a_n=1$

$$\lim_{n\to\infty} a_n^n = \lim_{n\to\infty}(1+a_n-1)^n = \lim_{n\to\infty}\left[(1+a_n-1)^{\frac{1}{a_n-1}}\right]^{(a_n-1)^n} =$$

$$= e^{\lim_{n\to\infty}(a_n-1)^n} = e^{\lim_{n\to\infty}\frac{f^{-1}\left(2+\frac{1}{n}\right)-f^{-1}(2)}{\frac{1}{n}}} \quad \textit{(3)}$$

$$\lim_{n\to\infty}\frac{f^{-1}\left(2+\frac{1}{n}\right)-f^{-1}(2)}{\frac{1}{n}} = (f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{2p+2} \quad \textit{(4)}$$

From (3)+ (4)$\Rightarrow \lim_{n\to\infty} a_n^n = e^{\frac{1}{2p+2}}$

Solution 2 by Remus Florin Stanca-Romania

$f(x) = x^{2p+1} + e^{x-1}; p \in \mathbb{N} \Rightarrow p > 0 \Rightarrow x^{2p+1}$ *is increasing;* $e > 1 \Rightarrow e^{x-1}$ *is increasing* $\Rightarrow$

$\Rightarrow e^{x-1} + x^{2p+1}$ *is increasing.*

$a_n = f^{-1}\left(2+\frac{1}{n}\right) \Rightarrow f(a_n) = 2 + \frac{1}{n} \Rightarrow a_n^{2p+1} + e^{a_n-1} = 2 + \frac{1}{n}(a); 2 + \frac{1}{n}$ *is decreasing,*

but $f(x) = x^{2p+1} + e^{x-1}$ *is increasing* $\Rightarrow a_n$ *is a decreasing sequence.*

We suppose that $a_n < 1 \Rightarrow a_n^{2p+1} < 1(1)(P > 0)$

$a_n < 1 \Rightarrow a_n - 1 < 0 \Rightarrow e^{a_n-1} < 1\ (2)$

$$\overline{}\ "+"$$

$\underset{(1);(2)}{\overset{"+"}{\Rightarrow}} a_n^{2p+1} + e^{a_n-1} < 2 \overset{(a)}{\Rightarrow} 2 + \frac{1}{n} < 2 \Rightarrow \frac{1}{n} < 0 \Rightarrow$ *false, because*

$n \in \mathbb{N}^*\ (n \neq 0) \Rightarrow$ *contradiction, so* $a_n \geq 1$

$a_n \geq 1$ *and* a_n *is decreasing* $\Rightarrow a_n$ *is a verged sequence* $\Rightarrow \exists l \in \mathbb{R}$ *such that* $\lim_{n\to\infty} a_n = l$

$a_n^{2p+1} + e^{a_n-1} = 2 + \frac{1}{n} \Rightarrow l^{2p+1} + e^{l-1} = 2; f(l) = l^{2p+1} + e^{l-1}$ *is an increasing*

function $\Rightarrow f$ *injective* $\Rightarrow$ *we have a unique solution* $\Rightarrow l \overset{unique}{=} 1 \Rightarrow \lim_{n\to\infty} a_n = 1$

$$\Omega = \lim_{n\to\infty}(a_n)^n = \lim_{n\to\infty}(a_n-1+1)^{\frac{1}{a_n-1}(a_n-1)n} = \lim_{n\to\infty} e^{\frac{a_n-1}{\frac{1}{n}}}$$

$$a_n^{2p+1} + e^{a_n-1} = 2 + \frac{1}{n} \Rightarrow \frac{1}{n} = a_n^{2p+1} + e^{a_n-1} - 2 \Rightarrow \Omega = \lim_{n\to\infty} e^{\frac{a_n-1}{a_n^{2p+1}+e^{a_n-1}}} \underset{\frac{0}{0}}{\overset{Stolz\ Cesaro}{=}}$$

$$= \lim_{n\to\infty} e^{\frac{a_{n+1}-a_n}{a_{n+1}^{2p+1}-a_n^{2p+1}+e^{a_{n+1}-1}-e^{a_n-1}}} = \lim_{n\to\infty} e^{\frac{1}{\frac{a_{n+1}^{2p+1}-a_n^{2p+1}}{a_{n+1}-a_n}+\frac{e^{a_{n+1}-1}-e^{a_n-1}}{a_{n+1}-a_n}}} =$$

$$= \lim_{n\to\infty} e^{\frac{\frac{1}{(a_{n+1}-a_n)\left(a_{n+1}^{2p}+a_{n+1}^{2p-1}a_n^1+\cdots+a_n^{2p}\right)}}{a_{n+1}-a_n}+e^{a_n-1}\cdot\frac{e^{a_{n+1}-a_n}-1}{a_{n+1}-a_n}} = e^{\frac{1}{2p+2}} \Rightarrow \Omega = \sqrt[2p+2]{e}$$

SP.190. Let be $x_0 > 0; x_{n+1} = x_n + \frac{1}{1+x_n+x_n^2+\cdots+x_n^p}; n \in \mathbb{N};$

$p \in \mathbb{N}; p \neq 0; p$ – fixed. Find:

$$\Omega = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{\sqrt[p+1]{n^{p+2}}}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Remus Florin Stanca-Romania

Let be $x_0 > 0; x_{n+1} = x_n + \frac{1}{1+x_n+x_n^2+\cdots+x_n^p}; n \in \mathbb{N}$

we prove by using the Mathematical induction that $x_n > 0; \forall n \in \mathbb{N}$*:*

1) we know that $P(0): x_0 > 0$ *is true.*

2) we suppose that $P(n): x_n > 0$ *is true.*

3) we prove that $P(n+1): x_{n+1} > 0$ *is true by using the fact that* $P(n)$ *is true:*

$$x_n > 0 \Rightarrow 1 + x_n + \cdots + x_n^p > 0 \Rightarrow \frac{1}{1 + x_n + \cdots + x_n^p} > 0$$

$$x_n > 0$$

$$\text{— — — — — — — — — — — — — — — — “+”}$$

$$\Rightarrow x_n + \frac{1}{1+x_n+\cdots+x_n^p} > 0 \Rightarrow x_{n+1} > 0 \Rightarrow P(n+1) \text{ is true} \Rightarrow$$

$$\succ x_n > 0\ \forall n \in \mathbb{N} \quad \textit{(Proved)}$$

$$x_{n+1} = x_n + \frac{1}{1 + x_n + \cdots + x_n^p} \Rightarrow x_{n+1} - x_n = \frac{1}{1 + x_n + \cdots + x_n^p} > 0 \Rightarrow x_{n+1} > x_n \Rightarrow$$

$\succ (x_n)_{n\in\mathbb{N}}$ *is an increasing sequence (1)*

We suppose that x_n *is verged* $\succ |\alpha \in R$ *such that* $x_n \leq \alpha$ *(2)*

$$\overset{(1);(2)}{\Rightarrow} \exists l \in \mathbb{R} \text{ such that } \lim_{n\to\infty} x_n = l; x_{n+1} = x_n + \frac{1}{1+\cdots+x_n^p} \Rightarrow$$

$$\Rightarrow l = l + \frac{1}{1+l+\cdots+l^p} \Rightarrow \frac{1}{1+\cdots+l^p}, \text{ false because } l \in \mathbb{R} \setminus \{+\infty; -\infty\}$$

$$\Rightarrow l = \infty \Rightarrow \lim_{n\to\infty} x_n = \infty$$

$$\Omega = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{\sqrt[p+1]{n^{p+2}}} \overset{Stolz\ Cesaro}{\underset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)^{\frac{p+2}{p+1}} - n^{\frac{p+2}{p+1}}} =$$

$$= \lim_{n\to\infty} \frac{x_{n+1}}{n^{\frac{p+2}{p+1}}\left(\left(\frac{n+1}{n}\right)^{\frac{p+2}{p+1}} - 1\right)} = \lim_{n\to\infty} \frac{x_{n+1}}{n^{\frac{p+2}{p+1}} \frac{\left(\frac{n+1}{n}\right)^{p+2} - 1}{\left(\left(\frac{n+1}{n}\right)^{\frac{p+2}{p+1}}\right)^0 + \cdots + \left(\left(\frac{n+1}{n}\right)^{\frac{p+2}{p+1}}\right)^p}} =$$

$$= \lim_{n\to\infty} \frac{x_{n+1}}{\frac{n^{\frac{p+2}{p+1}} \cdot \frac{1}{n}\left(\left(\frac{n+1}{n}\right)^0 + \left(\frac{n+1}{n}\right)^{p+1}\right)}{\left(\left(\frac{n+1}{n}\right)^{\frac{p+2}{p+1}}\right)^0 + \cdots + \left(\left(\frac{n+1}{n}\right)^{\frac{p+2}{p+1}}\right)^p}}$$

$$= \frac{p+1}{p+2} \cdot \lim_{n\to\infty} \frac{x_{n+1}}{n^{\frac{1}{p+1}}} = \frac{p+1}{p+2} \cdot \lim_{n\to\infty} \left(\frac{x_{n+1}^{p+1}}{n}\right)^{\frac{1}{p+1}} \quad \textbf{(3)}$$

$$x_{n+1} = x_n + \frac{1}{x_n^0 + \cdots + x_n^p} = x_n + \frac{x_n - 1}{x_n^{p+1} - 1} = \frac{x_n^{p+2} - 1}{x_n^{p+1} - 1} = x_{n+1} \quad \textbf{(4)}$$

$$\overset{(3)}{\succ} \Omega \overset{Stolz\ Cesaro}{\underset{\frac{\infty}{\infty}}{=}} \frac{p+1}{p+2} \lim_{n\to\infty} \left(x_{n+2}^{p+1} - x_{n+1}^{p+1}\right)^{\frac{1}{p+1}} = \frac{p+1}{p+2} \cdot \lim_{n\to\infty} \left(x_{n+1}^{p+1}\left(\left(\frac{x_{n+2}}{x_{n+1}}\right)^{p+1} - 1\right)\right)^{\frac{1}{p+1}} =$$

$$= \frac{p+1}{p+2} \lim_{n\to\infty} \left(x_{n+1}^{p+1} \cdot \frac{x_{n+2} - x_{n+1}}{x_{n+1}} \cdot (p+1)\right)^{\frac{1}{p+1}} =$$

$$= \frac{p+1}{p+2} \cdot \sqrt[p+1]{p+1} \cdot \lim_{n\to\infty} \left(x_{n+1}^p \cdot \frac{1}{x_{n+1}^0 + \cdots + x_{n+1}^p}\right)^{\frac{1}{p+1}} =$$

$$= \frac{p+1}{p+2} \cdot \sqrt[p+1]{p+1} \cdot 1^{\frac{1}{p+1}} \Rightarrow \Omega = \frac{p+1}{p+2} \cdot \sqrt[p+1]{p+1}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$x_{n+1} = x_n + \frac{1}{\sum_{m=0}^{p} x_n^m}$ ***and*** $x_0 > 0$ ***hence*** $x_1 > x_0, x_2 > x_1, \ldots, x_{n+1} > x_n$ ***for all***

$n \in \mathbb{N}$. Hence $\{x_n\}_{n=1}^{\infty}$ is an increasing function. Hence its bounded above and converges. Let $\lim_{n\to\infty} x_n = l$ then $l = l + \frac{1}{1+l+l^2+\cdots+l^p} \Rightarrow l \to \infty$, hence it is a contradiction. So, $\lim_{n\to\infty} x_n = \infty$. $\lim_{u\to 0}\frac{(1+u)^r-1}{u} = r$ where $r \in \mathbb{R}$

$$\Omega = \frac{x_1 + x_2 + \cdots + x_n}{\sqrt[p+1]{n^{p+2}}} \overset{\substack{CAESARO\\STOLZ}}{=} \frac{x_{n+1}}{(n+1)^{\frac{p+2}{p+1}} - n^{\frac{p+2}{p+1}}} =$$

$$= \left(\lim_{n\to\infty}\frac{1}{\frac{\left(1+\frac{1}{n}\right)^{\frac{p+2}{p+1}}-1}{\frac{1}{n}}}\right)\left(\lim_{n\to\infty}\frac{x_{n+1}}{\sqrt[p+1]{n}}\right)$$

$$= \frac{p+1}{p+2}\lim_{n\to\infty}\sqrt[p+1]{\frac{x_{n+1}^{p+1}}{n}} \overset{\substack{CAESARO\\STOLZ}}{=} \frac{p+1}{p+2}\lim_{n\to\infty}\sqrt[p+1]{\frac{x_{n+2}^{p+1}-x_{n+1}^{p+1}}{n+1-n}}$$

$$= \frac{p+1}{p+2}\lim_{n\to\infty}\sqrt[p+1]{\left(x_{n+1}+\frac{1}{\sum_{m=0}^{p}x_{n+1}^{m}}\right)^{p+1} - x_{n+1}^{p+1}}$$

$$= \frac{p+1}{p+2}\lim_{x_n\to 0}\sqrt[p+1]{\frac{x_{n+1}^{p}}{\sum_{m=0}^{p}x_{n+1}^{p}}\cdot\frac{\left(1+\frac{1}{x_{n+1}\left(\sum_{m=0}^{p}x_{n+1}^{m}\right)}\right)^{p+1}}{\frac{1}{\left(\sum_{m=0}^{p}x_{n+1}^{m}\right)}}} = \frac{p+1}{p+2}\sqrt[p+1]{p+1}$$

SP.191. Let be $f: [a,b] \to \mathbb{R}$; f – continuous and $\int_a^b f(x)\,dx = 0$.

Prove that exists $c \in (a,b)$ such that:

$$c \cdot f(c) \cdot \int_a^c f(t)\,dt = c \cdot f(c) + \int_a^c f(t)\,dt$$

Proposed by Marian Ursărescu – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\text{Let } g(x) = x\int_a^x f(t)\,dt \cdot e^{-\int_a^x f(t)dt} \quad (x \in [a,b])$$

$$\Rightarrow g'(x) = e^{-\int_a^x f(t)dt}\left[xf(x) + \int_a^x f(t)\,dt - xf(x)\int_a^x f(t)\,dt\right]$$

We have: $g(a) = g(b) = 0$, *by Rolle's theorem exists* $c \in (a,b)$ *such that:* $g'(c) = 0$

$$\Leftrightarrow e^{-\int_a^c f(t)dt}\left[cf(c) + \int_a^c f(t)\,dt - cf(c)\int_a^c f(t)dt\right] = 0$$

$$\Leftrightarrow cf(c)\int_a^c f(t)dt = cf(c) + \int_a^c f(t)\,dt\text{. Proved}$$

SP.192. Let be $A \in M_4(\mathbb{R})$; $\det A = 1$; $\det(A^2 + I_n) = 0$. Prove that:

$$\mathrm{Tr}(A^{-1}) = \mathrm{Tr}\,A$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$\text{As } \det(A^2 + I_4) = 0 \Rightarrow \det[(A + iA_4)(A - iI_4)] = 0$$

$$\Rightarrow \det(A + iI_4)\det(A - iI_4) = 0 \Rightarrow \det(A + iI_4) = 0 \text{ or } \det(A - iI_4) = 0$$

$\Rightarrow i$ *or* $-i$ *is an eigenvalue of* A. *As* $A \in M_4(\mathbb{R})$, *both* $i, -i$ *are eigenvalues of* A

Let λ, μ *be other eigenvalues of* A, *then* $1 = \det(A) = i(-i)\lambda\mu = \lambda\mu$

$$\Rightarrow \lambda\mu = 1 \Rightarrow \mu = \frac{1}{\lambda} \therefore \mathrm{Tr}(A) = i + (-i) + \lambda + \frac{1}{\lambda} = \lambda + \frac{1}{\lambda}$$

Also, $\mathrm{Tr}(A^{-1}) = \frac{1}{i} + \frac{1}{(-i)} + \frac{1}{\lambda} + \lambda = \frac{1}{\lambda} + \lambda$. *Thus,* $\mathrm{Tr}(A^{-1}) = \mathrm{Tr}(A)$

SP.193. If $A, B, C \in M_2(\mathbb{R})$; $\det A, \det B, \det C > 0$; $\det(ABC) = 8$ then:

$$\det(A^2 + B^2 + C^2) + \det(A^2 + B^2 - C^2) + \det(A^2 - B^2 + C^2) +$$
$$+ \det(-A^2 + B^2 + C^2) \geq 48$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

We use the theorem: If $M, N \in M_2(\mathbb{C})$ then:

$$P(x) = \det(M + Nx) = \det M + ax + \det N x^2$$

Let $p_1(x) = \det(A^2 + B^2 + C^2x) = \det(A^2 + B^2) + a_1x + \det C^2 x^2$

$$\left.\begin{array}{l} P_1(1) = \det(A^2 + B^2 + C^2) = \det(A^2 + B^2) + a_1 + \det C^2 \\ P_1(-1) = \det(A^2 + B^2 - C^2) = \det(A^2 + B^2) - a_1 + \det C^2 \end{array}\right\} \Rightarrow$$

$$\det(A^2 + B^2 + C^2) + \det(A^2 + B^2 - C^2) = 2(\det(A^2 + B^2) + \det C^2) \quad (1)$$

Let $P_2(x) = \det(C^2 + (A^2 - B^2)x) = \det C^2 + a_2x + \det(A^2 - B^2)x^2$

$$\left.\begin{array}{l} P_2(1) = \det(C^2 + A^2 - B^2) = \det C^2 + a_2 + \det(A^2 - B^2) \\ P_2(-1) = \det(C^2 - A^2 + B^2) = \det C^2 - a_2 + \det(A^2 - B^2) \end{array}\right\} \Rightarrow$$

$$\det(C^2 + A^2 - B^2) + \det(C^2 - A^2 + B^2) = 2(\det C^2 + \det(A^2 - B^2)) \quad (2)$$

From (1)+(2)$\Rightarrow \det(A^2 + B^2 + C^2) + \det(A^2 + B^2 - C^2) +$

$$+\det(A^2 - B^2 + C^2) + \det(A^2 + B^2 + C^2) = 4\det C^2 + 2\left(\det(A^2 + B^2) + \det(A^2 - B^2)\right)$$

Let $P_3(x) = \det(A^2 + xB^2) = \det A^2 + a_3x + \det B^2 x^2$ *(3)*

$$\left.\begin{array}{l} P_3(1) = \det(A^2 + B^2) = \det A^2 + a_3 + \det B^2 \\ P_3(-1) = \det(A^2 - B^2) = \det A^2 - a_3 + \det B^2 \end{array}\right\} \Rightarrow$$

$$\det(A^2 + B^2) + \det(A^2 - B^2) = 2(\det A^2 + \det B^2) \quad (4)$$

From (3)+(4)$\Rightarrow \det(A^2 + B^2 + C^2) + \det(A^2 + B^2 - C^2) +$

$$+\det(A^2 - B^2 + C^2) + \det(-A^2 + B^2 + C^2) = 4(\det A^2 + \det B^2 + \det C^2) \quad (5)$$

But $\det A^2 + \det B^2 + \det C^2 \geq 3\sqrt[3]{(\det A + \det B + \det C)^2} =$

$$= 3\sqrt[3]{(\det(ABC))^2} = 3\sqrt[3]{64} = 12 \quad (6)$$

From (5)+(6)$\Rightarrow \det(A^2 + B^2 + C^2) + \det(A^2 + B^2 - C^2) +$

$$+\det(A^2 - B^2 + C^2) + \det(-A^2 + B^2 + C^2) \geq 48$$

SP.194. Find all continuous functions $f: \mathbb{R} \to (0, +\infty)$ having the property:

$$f(x) \cdot f(ax) \cdot f(a^2x) = a^x, \forall x \in \mathbb{R}, a \in (0,1) \text{ – fixed.}$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$f(x)f(ax)f(a^2x)=a^x \quad \forall x\in\mathbb{R} \quad (1)$$

Put $x=0$, $f(0)^3=1\Rightarrow f(0)=1$. ***Replacing*** x ***by*** ax

$$f(ax)f(a^2x)f(a^3x)=a^{ax} \quad (2)$$

From (1), (2), we get: $\frac{f(x)}{f(a^3x)}=\frac{a^x}{a^{ax}}=a^{x(1-a)}$ ***(*)***

$$\frac{f(a^3x)}{f(a^3a^3x)}=a^{a^3x(1-a)}$$

[Replacing x ***by*** a^2x***]***

$$\Rightarrow \frac{f(a^3x)}{f(a^6x)}=a^{a^3x(1-a)} \quad (*)$$

$$\Rightarrow \frac{f(a^6x)}{f(a^9x)}=a^{a^6x(1-a)} \quad (*)$$

$$\vdots$$

$$\frac{f(a^{3n-3}x)}{f(a^{3n}x)}=a^{a^{3n-3}x(1-a)} \quad (*)$$

Multiplying n ***equations marked with (*), we get***

$$\frac{f(x)}{f(a^{3n}x)}=a^{x(1-a)[1+a^3+\dots+a^{3n-3}]}=a^{x(1-a)(1-a^{3n})/(1-a^3)}$$

Taking limit as $n\to\infty$

$$\left(a^{3n}\to 0 \text{ as } n\to\infty\ (\because 0<a<1)\right)$$

we obtain using continuity of f ***that***

$$\frac{f(x)}{f(0)}=a^{x(1-a)/(1-a^3)}=a^{x/(1+a+a^2)}\Rightarrow f(x)=a^{x/(1+a+a^2)}$$

SP.195. Find:

$$\lim_{n\to\infty}\sqrt[n]{\lim_{x\to 0}\frac{(e^x-\cos x)(e^{2x}-\cos 2x)\cdot\ldots\cdot(e^{nx}-\cos nx)-n!\,x^n}{\sin^{n+1}(n+1)x}}$$

Proposed by Marian Ursărescu – Romania

Solution by Remus Florin Stanca – Romania

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\lim_{x\to 0} \frac{(e^x - \cos x)(e^{2x} - \cos 2x)\cdot \ldots \cdot (e^{nx} - \cos(nx)) - n!\cdot x^n}{\sin^{(n+1)}\left((n+1)x\right)}}$$

$$\lim_{x\to 0} \frac{(e^x - \cos x)\cdot \ldots \cdot (e^{nx} - \cos(nx)) - n!\, x^n}{\sin^{(n+1)}\left((n+1)x\right)} \cdot \frac{\left((n+1)x\right)^{n+1}}{\left((n+1)x\right)^{n+1}} =$$

$$= \lim_{x\to 0} \frac{(e^x - \cos x)\cdot \ldots \cdot (e^{nx} - \cos nx) - n!\, x^n}{\left((n+1)x\right)^{n+1}} =$$

$$= \lim_{x\to 0} \frac{1}{(n+1)^{n+1}} \cdot \frac{(e^x - \cos x)\cdot \ldots \cdot (e^{nx} - \cos nx) - n!\, x^n}{x^{n+1}} =$$

Let $x_n = \lim_{x\to 0} \frac{(e^x - \cos x)\cdot \ldots \cdot (e^{nx} - \cos nx) - n! x^n}{x^{n+1}}$

$$\succ x_{n+1} = \lim_{x\to 0} \frac{(e^x - \cos x)\cdot \ldots \cdot \left(e^{(n+1)x} - \cos(n+1)x\right) - (n+1)!\, x^{n+1}}{x^{n+2}} =$$

$$= x_n \cdot \lim_{x\to 0} \frac{e^{(n+1)x} - \cos(n+1)x}{x} + \lim_{x\to 0} \frac{n! x^n \left(e^{(n+1)x} - \cos(n+1)x\right)}{x^{n+2}} - \lim_{x\to 0} \frac{(n+1)! x^{n+1}}{x^{n+2}} \quad \textbf{\textit{(1)}}$$

$$\lim_{x\to 0} \frac{e^{(n+1)x} - \cos(n+1)x}{x} = \lim_{x\to 0} \frac{e^{(n+1)x} - 1}{x} + \lim_{x\to 0} \frac{1 - \cos(n+1)x}{x} = n + 1 +$$

$$+ \lim_{x\to 0} \frac{1 - \cos(n+1)x}{x} \underset{\frac{0}{0}}{\overset{L'H}{=}} \lim_{x\to 0} \frac{(n+1)\sin(n+1)x}{1} + n + 1 = n + 1$$

$$\succ x_{n+1} = (n+1)x_n + n! \cdot \left(\lim_{x\to 0} \frac{e^{(n+1)x} - \cos\left((n+1)x\right) - (n+1)x}{x^2}\right)$$

$$\lim_{x\to 0} \frac{e^{(n+1)x} - \cos(n+1)\, x - (n+1)x}{x^2} \underset{\frac{0}{0}}{\overset{L'H}{=}}$$

$$= \lim_{x\to 0} \frac{e^{(n+1)x}(n+1) + (n+1)\sin\left((n+1)x\right) - (n+1)}{2x} =$$

$$= (n+1)\lim_{x\to 0} \frac{e^{(n+1)x} + \sin\left((n+1)x\right) - 1}{2x} = (n+1)\lim_{x\to 0} \frac{1}{2} \cdot \frac{e^{(n+1)x} - 1}{x} + \frac{1}{2} \cdot$$

$$\cdot \lim_{x\to 0} \frac{\sin\left((n+1)x\right)}{(n+1)x}(n+1) = (n+1)\left(\frac{n+1}{2} + \frac{n+1}{2}\right) = (n+1)^2 \Rightarrow$$

$$\Rightarrow x_{n+1} = (n+1)x_n + n!\,(n+1)(n+1) = (n+1)x_n + (n+1)!\,(n+1) = x_{n+1}$$

$$x_1 = \lim_{x\to 0} \frac{e^x - \cos x - x}{x^2} \underset{\frac{0}{0}}{\overset{L'H}{=}} \lim_{x\to 0} \frac{e^x + \sin x - 1}{2x} = 1 \Rightarrow x_1 = 1 \quad \textbf{\textit{(3)}}$$

We prove by using the Mathematical induction that: $x_n = n!\frac{n(n+1)}{2} = \frac{n(n+1)!}{2}$

1. we prove that $P(1): x_1 = \frac{1\cdot(1+1)!}{2}$ *is true*

$$P(1) \Leftrightarrow x_1 = \frac{1\cdot 2!}{2} = 1 \overset{(3)}{\Rightarrow} P(1) \text{ is true}$$

2. we suppose that $P(n): x_n = \frac{n(n+1)!}{2}$ *is true*

3. we prove that $P(n+1): x_{n+1} = \frac{(n+1)(n+2)!}{2}$ *is true by using the fact that* $P(n)$ *is true:*

$$x_{n+1} = (n+1)x_n + (n+1)!(n+1) \Leftrightarrow x_{n+1} = (n+1)\cdot\frac{n(n+1)!}{2} + (n+1)!(n+1) =$$

$$= (n+1)!(n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)!(n+1)(n+2)}{2} = \frac{(n+2)!(n+1)}{2} \Rightarrow$$

$$\Rightarrow P(n+1) \text{ is true} \Rightarrow x_n = \frac{n(n+1)!}{2} \quad \forall n \in \mathbb{N}^* \Rightarrow \text{proved}$$

$$\succ \lim_{x\to 0}\frac{(e^x - \cos x)\cdot\ldots\cdot(e^{nx} - \cos(nx)) - n!\,x^n}{x^{n+1}} = \frac{n(n+1)!}{2}$$

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\frac{x_n}{(n+1)^{n+1}}} = \lim_{n\to\infty} e^{\frac{\ln\frac{x_n}{(n+1)^{n+1}}}{n}} \overset{Stolz\ Cesaro}{=} \lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\cdot\left(\frac{n+1}{n+2}\right)^{(n+1)}\cdot\frac{1}{n+2}\right) =$$

$$= \frac{1}{e}\cdot\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\cdot\frac{1}{n+2}\right) = \frac{1}{e}\cdot\frac{(n+1)(n+2)!}{n(n+1)!}\cdot\frac{1}{n+2} = \frac{1}{e}\cdot\lim_{n\to\infty}\frac{n+1}{n} = \frac{1}{e} \succ \Omega = \frac{1}{e}$$

UP.181. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \left(e^{\sin^2 x + \frac{\sin^4 x}{2\cos^2 x}}\right)dx \ge \tan b - \tan a$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$e^{\sin^2 x + \frac{\sin^4 x}{2\cos^2 x}} \ge \frac{1}{\cos^2 x}\left(0 < x < \frac{\pi}{2}\right)$$

$$\Leftrightarrow e^{(1-\cos^2 x) + \frac{(1-\cos^2 x)^2}{2\cos^2 x}} \ge \frac{1}{\cos^2 x} \quad (*)$$

Let $t = \cos^2 x \ \ (0 < t < 1) \because f(t) = e^{(1-t)+\frac{(1-t)^2}{2t}} - \frac{1}{t} \ \ (0 < t < 1)$

$$\Rightarrow f'(t) = -\frac{t^2+1}{2t^2} \cdot e^{(1-t)+\frac{(1-t)^2}{2t}} + \frac{1}{t^2} = \frac{2 - \left[(t^2+1)e^{(1-t)+\frac{(1-t)^2}{2t}}\right]}{2t^2} < 0$$

(∵ Because: $(t^2+1) \cdot e^{(1-t)+\frac{(1-t)^2}{2t}} > 2 \ \ (0 < t < 1)$ ***)***

$$\Rightarrow f(t) \searrow (0;1) \Rightarrow f(t) > f(1) = 0 \Rightarrow \textit{(*) true.}$$

Hence: $\int_a^b e^{\sin^2 x + \frac{\sin^4 x}{2\cos^2 x}} dx \ge \int_a^b \frac{1}{\cos^2 x} dx = \tan b - \tan a$. ***Proved.***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$e^x \ge 1 + x$ ***for all*** $x \ge 0$ ***then:***

$$\int_a^b e^{\sin^2 x + \frac{\sin^4 x}{2\cos^2 x}} dx = \int_a^b e^{\sin^2 x\left(1+\frac{\tan^2 x}{2}\right)} dx = \int_a^b e^{\sin^2 x\left(\frac{1+\sec^2 x}{2}\right)} dx$$

$$\overset{AM\ge GM}{\ge} \int_a^b e^{\sec x \cdot \sin^2 x} dx = \int_a^b e^{\sec x - \cos x} dx = \int_a^b e^{\frac{\sec^2 x - 1}{\cos x}} dx$$

$$\ge \int_a^b e^{\sec^2 x - 1} dx \ \ [since, 1 \ge \cos x \ge -1] \ge \int_a^b \sec^2 x \, dx = \tan b - \tan a$$

(proved)

UP.182. Find:

$$\int_0^1 x^2 \ln^2(x) \left\{\frac{1}{x}\right\} dx$$

where $\{\cdot\}$ denotes the Fractional Part.

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Zaharia Burghelea-Romania

Denote: $\Omega = \int_0^1 x^2 \ln^2 x \left\{\frac{1}{x}\right\} dx$

Consider: $I(a)=\int_0^1 x^a\left\{\frac{1}{x}\right\}dx\overset{x=\frac{1}{t}}{=}\int_1^\infty\frac{\{t\}}{t^{a+2}}dt=$

$$=\sum_{n=1}^{\infty}\int_n^{n+1}\frac{(t-n)}{t^{a+2}}dt\overset{t-n=y}{=}\sum_{n=1}^{\infty}\int_0^1\frac{y}{(y+n)^{a+2}}dy$$

$$=\sum_{n=1}^{\infty}\int_0^1\left(\frac{1}{(y+n)^{a+1}}-\frac{n}{(y+n)^{a+2}}\right)dy=\sum_{n=1}^{\infty}\left(-\frac{1}{a}\cdot\frac{1}{(y+n)^a}+\frac{n}{a+1}\cdot\frac{1}{(y+n)^{a+1}}\right)\Big|_0^1$$

$$=\frac{1}{a}\sum_{n=1}^{\infty}\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)-\frac{1}{a+1}\sum_{n=1}^{\infty}\left(\frac{n}{n^{a+1}}-\frac{n}{(n+1)^{a+1}}\right)=\frac{1}{a}-\frac{\zeta(a+1)}{a+1}$$

$$\Omega=\int_0^1 x^2\ln^2 x\left\{\frac{1}{x}\right\}dx=\frac{d^2}{da^2}(I(a))\Big|_{a=2}$$

Using: $\frac{d}{ds}\zeta(s)=\sum_{n=1}^{\infty}\frac{d}{ds}(n^{-s})=-\sum_{n=2}^{\infty}\frac{\ln n}{n^s}=-\zeta'(s)$

$$\Rightarrow\frac{d^2}{da^2}(I(a))=\frac{2}{a^3}-\frac{2\zeta(a+1)}{(a+1)^3}+\frac{2\zeta'(a+1)}{(a+1)^2}-\frac{\zeta''(a+1)}{a+1}$$

$$\Rightarrow\Omega=\frac{d^2}{da^2}(I(a))\Big|_{a=2}=\int_0^1 x^2\ln^2 x\left\{\frac{1}{x}\right\}dx=\frac{1}{4}-\frac{2}{27}\zeta(3)+\frac{2}{9}\zeta'(3)-\frac{1}{3}\zeta''(3)$$

Solution 2 by Tobi Joshua-Nigeria

$$I=\int_0^1 x^2\ln^2 x\left\{\frac{1}{x}\right\}dx,\ t=\frac{1}{x};\ I=\int_1^\infty\frac{\ln^2 t}{t^4}\{t\}dt=\int_1^\infty\frac{\ln^2 t}{t^4}\ (t-[t])dt$$

$$I=\int_1^\infty\frac{t\cdot\ln^2 t\,dt}{t^4}-\int_1^\infty\frac{\ln^2 t\,[t]dt}{t^4};\ I=\frac{1}{4}-\int_1^\infty[t]\frac{\ln^2 t}{t^4}dt$$

$$I=\frac{1}{4}-\sum_{k=1}^{\infty}k\int_k^{k+1}\frac{\ln^2 t}{t^4}dt;\ I=\frac{1}{4}-\sum_{k=1}^{\infty}k\left[-\frac{2}{27t^3}-\frac{\ln^2 t}{3t^3}-\frac{2\ln t}{9t^3}\right]_k^{k+1}$$

$$I=\frac{1}{4}+\sum_{k=1}^{\infty}k\left[\frac{+2}{27t^3}+\frac{\ln^2 t}{3t^3}+\frac{2\ln t}{at^3}\right]_k^{k+1}$$

$$I=\frac{1}{4}+\sum_{k=1}^{\infty}k\left(\frac{0}{27(k+1)^3}-\frac{2}{27k^3}\right)+\frac{1}{3}\sum_{k=1}^{\infty}k\left(\frac{\ln^2(k+1)}{(k+1)^3}+\frac{\ln^2 k}{k^3}\right)+\frac{2}{9}$$

$$\sum_{k=1}^{\infty} k\left(\frac{\ln(k+1)}{(k+1)^3}-\frac{\ln k}{k^3}\right)$$

$$I=\frac{1}{4}-\frac{2}{27}I(3)+\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{k\ln^2(k+1)}{(k+1)^3}-\frac{\ln k}{k^2}\right)+\frac{2}{4}$$

$$\sum_{k=1}^{\infty}\left(\frac{k\ln(k+1)}{(k+1)^3}-\frac{\ln k}{k^2}\right)$$

$$A=\sum_{k=1}^{\infty}\left(\frac{k\ln^2(k+1)}{(k+1)^3}-\frac{\ln^2 k}{k^2}\right)$$

$$A=\sum_{k=1}^{\infty}\frac{\partial^2}{\partial a^2}|a=3\frac{k}{(k+1)^9}-\sum_{k=1}^{\infty}\frac{\partial^2}{\partial a^2}|a=2\frac{1}{k^9}$$

$$A=\frac{\partial^2}{\partial a^2}|a=3\big(\zeta(a-1)-\zeta(a)\big)-\frac{\partial^2}{\partial a^2}|a=2\,\zeta(a)$$

$$A=\zeta''(2)-\zeta''(3)-\zeta''(2)=-\zeta''(3)\quad \textbf{(1)}$$

$$B=\sum_{k=1}^{\infty}\left(\frac{k\ln(k+1)}{(k+1)^3}-\frac{\ln k}{k^2}\right)$$

$$B=\sum_{k=1}^{\infty}-\frac{\partial}{\partial a}|a=3\frac{k}{(k+1)^a}+\sum_{k}^{\infty}+\frac{\partial}{\partial a}|a=2\frac{1}{k^a}$$

$$B=-\frac{\partial}{\partial a}|a=3\big(\zeta(a-1)-\zeta(a)\big)+\frac{\partial}{\partial a}|a=2\zeta(a)$$

$$B=-\zeta'(2)+\zeta'(3)+\zeta'(2)=+\zeta'(3)\quad \textbf{(2)}$$

$$I=\frac{1}{4}-\frac{2}{27}\zeta(3)-\frac{\zeta''(3)}{3}+\frac{27\zeta'(3)}{9}$$

$$I=\frac{1}{4}-\frac{2}{27}\zeta(3)-\frac{\zeta''(3)}{3}+\frac{2}{9}\zeta'(3)$$

Solution 3 by Kartick Chandra Betal-India

$$\int_0^1 x^2\ln^2 x\left\{\frac{1}{x}\right\}dx=\int_1^{\infty}\frac{\ln^2 x}{x^4}\{x\}dx$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n-1}\int_k^{k+1}\frac{(x-k)}{x^4}\cdot\ln^2 x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n-1}\int_k^{k+1}\left(\frac{\ln^2 x}{x^3}-\frac{k\ln^2 x}{x^4}\right)dx$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n-1}\left[-\frac{\ln^2 x}{2x^2}-\frac{\ln x}{2x^2}-\frac{1}{4x^2}+k\left(\frac{\ln^2 x}{3x^3}+\frac{2\ln x}{9x^3}+\frac{2}{27x^3}\right)\right]_k^{k+1}$$

$$= \lim_{n\to\infty}\left[\left\{-\frac{\ln^2 n}{2n^2}-\frac{\ln x}{2n^2}-\frac{1}{4}\left(\frac{1}{n^2}-1\right)\right\}+\left\{\frac{\ln^2 n}{3n^2}+\frac{2\ln n}{9n^2}+\frac{2}{27}\left(\frac{1}{n^2}-1\right)\right\}- -\sum_{k=1}^{n-1}\left\{\frac{\ln^2(1+k)}{3(1+k)^3}+\frac{2\ln(1+k)}{9(1+k)^3}+\frac{2}{27(1+k)^3}\right\}\right]$$

$$= \lim_{n\to\infty}\left[\left\{-\frac{\ln^2 n}{6n^2}-\frac{5\ln n}{18n^2}-\frac{19}{108n^2}+\frac{19}{108}\right\}-\sum_{k=1}^{n}\left\{\frac{\ln^2 k}{3k^3}+\frac{2\ln k}{9k^3}+\frac{2}{27}\left(\frac{1}{k^3}-1\right)\right\}\right]$$

$$= \frac{19}{108}+\frac{2}{27}-\frac{\zeta''(3)}{3}+\frac{2\zeta'(3)}{9}-\frac{2}{27}\zeta(3) = \frac{1}{4}-\frac{\zeta''(3)}{3}+\frac{2\zeta'(3)}{9}-\frac{2}{27}\zeta(3)$$

UP.183. Let x_n, y_n, z_n be three sequences of real numbers such that:

$$\lim_{n\to\infty}\frac{x_n}{n^p}=a,\ \lim_{n\to\infty}\frac{y_n}{n^{p+1}}=b,\ \lim_{n\to\infty}\frac{z_n}{n^{p+2}}=c,\ a,b,c\in\mathbb{R}^*,\ p\in\mathbb{N}^*$$

Find:

$$\lim_{n\to\infty}\frac{(x_1^3+\cdots+x_n^3)(y_1^3+\cdots+y_n^3)(z_1^3+\cdots+z_n^3)}{(x_1y_1z_1+\cdots+x_ny_nz_n)^3}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{u\to 0}\frac{(1+u)^r-1}{u}=r\in\mathbb{R}\text{, then }\lim_{n\to\infty}\frac{\sum_{k=1}^n x_k^3}{n^{3p+1}}\overset{CESARO\ STOLZ}{=}\lim_{n\to\infty}\frac{x_{n+1}^3}{(n+1)^{3p+1}-n^{3p+1}}$$

$$= \lim_{n\to\infty}\left(\frac{x_{n+1}}{n^p}\right)^3\cdot\lim_{n\to\infty}\frac{1}{\frac{\left(1+\frac{1}{n}\right)^{3p+1}-1}{\frac{1}{n}}} = \frac{a^3}{3p+1}$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}y_k^3}{n^{3p+4}}\overset{\substack{CESARO\\STOLZ}}{=}\lim_{n\to\infty}\frac{y_{n+1}^3}{(n+1)^{3p+4}-n^{3p+4}}=\lim_{n\to\infty}\left(\frac{y_n}{n^{p+1}}\right)^3\cdot\frac{1}{\frac{\left(1+\frac{1}{n}\right)^{3p+4}-1}{\frac{1}{n}}}$$

$$=\frac{b^3}{3p+4}\lim_{n\to\infty}\frac{\sum_{k=1}^{n}z_k^3}{n^{3p+7}}\overset{\substack{CESARO\\STOLZ}}{=}\lim_{n\to\infty}\frac{z_{n+1}^3}{(n+1)^{3p+7}-n^{3p+7}}$$

$$=\lim_{n\to\infty}\left(\frac{z_n}{n^{p+2}}\right)^3\cdot\lim_{n\to\infty}\frac{1}{\frac{\left(1+\frac{1}{n}\right)^{3p+7}-1}{\frac{1}{n}}}=\frac{c^3}{3p+7}$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}x_ky_kz_k}{n^{3p+4}}\overset{\substack{CESARO\\STOLZ}}{=}\lim_{n\to\infty}\frac{x_{n+1}y_{n+1}z_{n+1}}{(n+1)^{3p+4}-n^{3p+4}}=\frac{abc}{3p+4}$$

$$\lim_{n\to\infty}\left(\frac{(x_1^3+\cdots+x_n^3)(y_1^3+\cdots+y_n^3)(z_1^3+\cdots+z_n^3)}{(x_1y_1z_1+\cdots+x_ny_nz_n)^3}\right)=$$

$$=\lim_{n\to\infty}\frac{\left(\frac{\sum_{k=1}^{n}x_k^3}{n^{3p+1}}\right)\left(\frac{\sum_{k=1}^{n}y_k^3}{n^{3p+4}}\right)\left(\frac{\sum_{k=1}^{n}z_k^3}{n^{3p+7}}\right)}{\frac{\left(\sum_{k=1}^{n}x_ky_kz_k\right)^3}{n^{9p+12}}}$$

$$=\frac{\frac{a^3}{3p+1}\cdot\frac{b^3}{3p+4}\cdot\frac{c^3}{3p+7}}{\left(\frac{abc}{3p+4}\right)^3}=\frac{(3p+4)^2}{(3p+1)(3p+7)}\ \textit{(Answer)}$$

Solution 2 by Remus Florin Stanca-Romania

Let x_n, y_n, z_n be three sequences of real numbers such that:

$\lim_{n\to\infty}\frac{x_n}{n^p}=a, \lim_{n\to\infty}\frac{y_{n+1}}{n^{p+1}}=b, \lim_{n\to\infty}\frac{z_n}{n^{p+2}}=c, a,b,c\in\mathbb{R}^*, p\in\mathbb{N}^*$. ***Find:***

$$\Omega=\lim_{n\to\infty}\frac{(x_1^3+\cdots+x_n^3)(y_1^3+\cdots+y_n^3)(z_1^3+\cdots+z_n^3)}{(x_1y_1z_1+\cdots+x_ny_nz_n)^3}$$

$$\Omega=\lim_{n\to\infty}\frac{x_1^3+\cdots+x_n^3}{(n+1)^{3p+1}}\cdot\frac{y_1^3+\cdots+y_n^3}{(n+1)^{3p+4}}\cdot\frac{z_1^3+\cdots+z_n^3}{(n+1)^{3p+7}}\cdot\frac{(n+1)^{9p+12}}{(x_1y_1z_1+\cdots+x_ny_nz_n)^3}\quad (a)$$

$$\lim_{n\to\infty}\frac{x_1^3+\cdots+x_n^3}{(n+1)^{3p+1}}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{x_{n+1}^3}{(n+2)^{3p+1}-(n+1)^{3p+1}}=$$

$$=\lim_{n\to\infty}\frac{x_{n+1}^3}{(n+1)^{3p+1}\left(\left(\frac{n+2}{n+1}\right)^{3p+1}-1\right)}=$$

$$=\lim_{n\to\infty}\frac{x_{n+1}^3}{(n+1)^{3p+1}\left(\frac{n+2}{n+1}-1\right)\left(\left(\frac{n+2}{n+1}\right)^0+\cdots+\left(\frac{n+2}{n+1}\right)^{3p}\right)}$$

$$=\lim_{n\to\infty}\left(\frac{x_{n+1}}{(n+1)^p}\right)^3\cdot\frac{1}{3p+1}=\frac{a^3}{3p+1}\quad (1)$$

$$\lim_{n\to\infty}\frac{y_1^3+\cdots+y_n^3}{(n+1)^{3p+4}}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{y_{n+1}^3}{(n+2)^{3p+4}-(n+1)^{3p+4}}=$$

$$=\lim_{n\to\infty}\frac{y_{n+1}^3}{(n+1)^{3p+4}\cdot\frac{1}{n+1}\left(\left(\frac{n+2}{n+1}\right)^0+\cdots+\left(\frac{n+2}{n+1}\right)^{3p+3}\right)}$$

$$=\lim_{n\to\infty}\left(\frac{y_{n+1}}{(n+1)^{p+1}}\right)^3\cdot\frac{1}{3p+4}=\frac{b^3}{3p+4}\quad (2)$$

$$\lim_{n\to\infty}\frac{z_1^3+\cdots+z_n^3}{(n+1)^{3p+7}}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{z_{n+1}^3}{(n+2)^{3p+7}-(n+1)^{3p+7}}=$$

$$=\lim_{n\to\infty}\frac{z_{n+1}^3}{(n+1)^{3p+7}\cdot\frac{1}{n+1}\left(\left(\frac{n+2}{n+1}\right)^0+\cdots+\left(\frac{n+2}{n+1}\right)^{3p+6}\right)}=\frac{c^3}{3p+7}\quad (3)$$

$$\lim_{n\to\infty}\frac{(n+1)^{9p+12}}{(x_1y_1z_1+\cdots+x_ny_nz_n)^3}=\lim_{n\to\infty}\left(\frac{(n+1)^{3p+4}}{x_1y_1z_1+\cdots+x_ny_nz_n}\right)^3\overset{Stolz\ Cesaro}{=}$$

$$=\lim_{n\to\infty}\left(\frac{(n+2)^{3p+4}-(n+1)^{3p+4}}{x_{n+1}y_{n+1}z_{n+1}}\right)^3=$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)^{3p+4}\cdot\frac{1}{n+1}\left(\left(\frac{n+2}{n+1}\right)^0+\cdots+\left(\frac{n+2}{n+1}\right)^{3p+3}\right)}{x_{n+1}y_{n+1}z_{n+1}}\right)^3=$$

$$=(3p+4)^3\cdot\lim_{n\to\infty}\left(\frac{(n+1)^p}{x_{n+1}}\cdot\frac{(n+1)^{p+1}}{y_{n+1}}\cdot\frac{(n+1)^{p+2}}{z_{n+1}}\right)^3=$$

$$=(3p+4)^3\cdot\frac{1}{a^3b^3c^3}\quad (4)$$

$$\overset{(a);(1);(2);(3)}{\Rightarrow}\Omega=\frac{a^3b^3c^3}{(3p+1)(3p+4)(3p+7)}\cdot\frac{(3p+4)^3}{a^3b^3c^3}\Rightarrow\Omega=\frac{(3p+4)^2}{(3p+1)(3p+7)}$$

UP.184. If $0 < a \le b < \frac{\pi}{2}$ then:

$$4\int_a^b \left((\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)^3\right) dx \ge 125(b-a)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

We prove that: $4[(\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)^3] \ge 125$

$$\Leftrightarrow [(\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)^3] \ge \frac{125}{4} \quad (*)$$

$$\because LHS_{(*)} \ge \frac{[(\sin^2 x + \csc^2 x) + (\cos^2 x + \sec^2 x)]^3}{2^2}$$

$$= \frac{\left[1 + \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\right]^3}{2^2} = \frac{\left[1 + \frac{4}{\sin^2 2x}\right]^3}{2^2} \ge \frac{[1+4]^3}{2^2} = \frac{125}{4}$$

$$\Rightarrow 4\int_a^b [(\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)]\, dx \ge 125\int_a^b dx = 125(b-a)$$

(Proved)

Solution 2 by Avishek Mitra-West Bengal-India

$$S = (\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)^3$$

$$= (\sin^6 x + \csc^6 x) + 3(\sin^2 x + \cos^2 x) + 3(\sec^2 x + \csc^2 x) + (\sec^6 x + \csc^6 x)$$

$$\Rightarrow \sin^6 x + \cos^6 x = 1 - \frac{1}{3}(\sin 2x)^2 \Leftrightarrow (\sin 2x)^2 \le 1 \Rightarrow \sin^6 x + \cos^6 x \ge \frac{1}{4}$$

$$\Rightarrow \sec^2 x + \csc^2 x = \frac{1}{\sin^2 x \cdot \cos^2 x} = \frac{4}{(\sin 2x)^2} \Rightarrow (\sec^2 x + \csc^2 x) \ge 4$$

$$\Rightarrow \frac{\sec^2 x + \csc^6 x}{2} \overset{AM-GM}{\ge} \frac{1}{\sin^3 x \cdot \cos^3 x} \Rightarrow p \ge \frac{2}{\frac{1}{8}(\sin 2x)^2} \Leftrightarrow (\sin 2x)^2 \le 1 \Rightarrow p \ge 16$$

$$\Leftrightarrow S \ge \frac{1}{4} + 3 + (3 \times 4) + 16 \Rightarrow S \ge \frac{125}{4} \Rightarrow \int_a^b S\, dx = \frac{125}{4}\int_a^b dx$$

$$\Leftrightarrow 4\int_a^b ((\sin^2 x+\csc^2 x)^3+(\cos^2 x+\sec^2 x)^3)dx \ge 125(b-a)$$

(Proved)

UP.185. Calculate the integral:

$$\int_0^\infty \frac{x^2\arctan x}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Zaharia Burghelea-Romania

Denote: $I=\int_0^\infty \frac{x^2\arctan x}{x^4+x^2+1}dx$

$$I\overset{x=\frac{1}{t}}{=}\int_0^\infty \frac{\arctan\left(\frac{1}{t}\right)}{t^4+t^2+1}dt \Rightarrow 2I=\int_0^\infty \frac{x^2\arctan x+\frac{\pi}{2}-\arctan x}{x^4+x^2+1}dx$$

$$I=\frac{\pi}{4}\int_0^\infty \frac{dx}{x^2+x^2+1}+\frac{1}{2}\int_0^\infty \frac{(x^2-1)\arctan x}{x^4+x^2+1}dx$$

$$I_1=\int_0^\infty \frac{dx}{x^4+x^2+1}\overset{x=\frac{1}{t}}{=}\int_0^\infty \frac{t^2}{t^4+t^2+1}dt \Rightarrow 2I_1=\int_0^\infty \frac{x^2+1}{x^4+x^2+1}dx$$

$$I_1=\frac{1}{2}\int_{-\infty}^\infty \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+3}=\frac{1}{2\sqrt{3}}\arctan\left(\frac{x^2-1}{\sqrt{3}x}\right)\Big|_{-\infty}^{\infty}=\frac{\pi}{2\sqrt{3}}$$

$$\Rightarrow I=\frac{\pi^2}{8\sqrt{3}}+\frac{1}{2}\int_0^\infty \frac{(x^2-1)\arctan x}{x^4+x^2+1}dx=\frac{\pi^2}{8\sqrt{3}}+\frac{1}{2}I_2$$

$$I_2=\int_0^\infty \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1}\arctan x\,dx=\int_0^\infty \left(\frac{1}{2}\ln\left(\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right)\right)'\arctan x\,dx$$

$$=\frac{1}{2}\ln\left(\frac{x^2-x+1}{x^2+x+1}\right)\arctan x\Big|_0^\infty-\frac{1}{2}\int_0^\infty \ln\left(\frac{x^2-x+1}{x^2+x+1}\right)\frac{dx}{1+x^2}$$

$$= \frac{1}{2}\int_0^\infty \ln\left(\frac{x^2+x+1}{x^2-x+1}\right) \overset{x=\tan t}{=} \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\frac{\sec^2+\tan t}{\sec^2 t-\tan t}\right)dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\sin t\cos t}{1-\sin t\cos t}\right)dt \overset{2t=x}{=} \frac{1}{4}\int_0^{\pi} \ln\left(\frac{1+\frac{1}{2}\sin x}{1-\frac{1}{2}\sin x}\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\frac{1}{2}\sin x}{1-\frac{1}{2}\sin x}\right)dx$$

Consider the following integral: $I(a) = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right)dt$

$$I'(a) = \int_0^{\frac{\pi}{2}} \left(\frac{\sin x}{1+a\sin x}+\frac{\sin x}{1-a\sin x}\right)dx = 2\int_0^{\frac{\pi}{2}} \frac{\sin x}{1-a^2\sin^2 x}dx =$$

$$= 2\int_0^{\pi} \frac{\sin x}{a^2\cos^2 x+\left(\sqrt{1-a^2}\right)^2}dx = \frac{2}{a^2}\int_0^{1} \frac{d(\cos x)}{\cos^2 x+\left(\frac{\sqrt{1-a^2}}{a}\right)^2}dx$$

$$= \frac{2a}{a^2\sqrt{1-a^2}}\arctan\left(\frac{ax}{\sqrt{1-a^2}}\right)\Bigg|_0^1 = \frac{2}{a\sqrt{1-a^2}}\arctan\left(\frac{a}{\sqrt{1-a^2}}\right)$$

$$I(0) = 0 \Rightarrow I_2 = \frac{1}{2}I\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{1}{a\sqrt{1-a^2}}\arctan\left(\frac{a}{\sqrt{1-a^2}}\right)da \overset{a=\sin x}{=} \int_0^{\frac{\pi}{6}} \frac{x}{\sin x}dx =$$

$$= \int_0^{\frac{\pi}{6}} x\left(\ln\left(\tan\frac{x}{2}\right)\right)' dx = x\ln\left(\tan\frac{x}{2}\right)\Bigg|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \ln\left(\tan\frac{x}{2}\right)dx$$

$$\overset{\frac{x}{2}=t}{=} \frac{\pi}{6}\ln(2-\sqrt{3}) - 2\int_0^{\frac{\pi}{12}} \ln(\tan t)\,dt = -\frac{\pi}{6}\ln(2+\sqrt{3}) - 2I_3$$

Consider: $I_4 = \int_0^{\frac{\pi}{12}} \ln(\tan(3t))\,dt$ ***and use the following identity:***

$$\tan(3x) = \tan(3x)\tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right)$$

$$I_4 = \int_0^{\frac{\pi}{12}} \ln(\tan t)\,dt + \int_0^{\frac{\pi}{12}} \ln\left(\tan\left(\frac{\pi}{3}-t\right)\right)dt + \int_0^{\frac{\pi}{12}} \ln\left(\tan\left(\frac{\pi}{3}+t\right)\right)dt =$$

$$=\int_0^{\frac{\pi}{12}}\ln(\tan x)\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\ln(\tan x)\,dx+\int_{\frac{\pi}{3}}^{\frac{5\pi}{12}}\ln(\tan x)\,dx=$$

$$=\int_0^{\frac{\pi}{12}}\ln(\tan x)\,dx+\int_{\frac{\pi}{4}}^{\frac{5\pi}{12}}\ln(\tan x)\,dx=\int_0^{\frac{\pi}{12}}\ln(\tan x)\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{12}}\ln(\tan x)\,dx=$$

$$=2\int_0^{\frac{\pi}{12}}\ln(\tan x)\,dx-3\int_0^{\frac{\pi}{12}}\ln(\tan(3x))\,dx\Rightarrow I_4=2I_3-3I_4$$

$$\Rightarrow I_3=2I_4=2\int_0^{\frac{\pi}{12}}\ln(\tan(3x))\,dx=\frac{2}{3}\int_0^{\frac{\pi}{4}}\ln(\tan x)\,dx=\frac{2}{3}\int_0^1\frac{\ln t}{1+t^2}dt=$$

$$=\frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n}\ln t\,dt=\frac{2}{3}-\frac{2}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-\frac{2}{3}G$$

$$\Rightarrow I_2=\int_0^{\frac{\pi}{6}}\frac{x}{\sin x}dx=\frac{4}{3}G-\frac{\pi}{6}\ln\left(2+\sqrt{3}\right)$$

$$\Rightarrow I=\int_0^{\infty}\frac{x^2\arctan x}{x^4+x^2+1}dx=\frac{\pi^2}{8\sqrt{3}}-\frac{\pi}{12}\ln\left(2+\sqrt{3}\right)+\frac{2}{3}G$$

Solution 2 by Kartick Chandra Betal-India

$$I=\int_0^{\infty}\frac{x^2\tan^{-1}x}{x^4+x^2+1}dx=\int_0^{\infty}\frac{\cot^{-1}x}{x^4+x^2+1}dx=\int_0^{\infty}\frac{\frac{\pi}{2}-\tan^{-1}x}{x^4+x^2+1}dx$$

$$2I=\int_0^{\infty}\frac{\frac{\pi}{2}+(x^2-1)\tan^{-1}x}{x^4+x^2+1}dx=\frac{\pi}{2}\int_0^{\infty}\frac{dx}{x^4+x^2+1}+\int_0^{\infty}\frac{\left(1-\frac{1}{x^2}\right)\cdot\tan^{-1}x}{x^4+x^2+1}dx$$

$$=\frac{\pi}{4}\int_0^{\infty}\frac{1+\frac{1}{x^2}-\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}+1}dx+\int_0^{\infty}\frac{\frac{d}{dx}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-1}\tan^{-1}x\,dx$$

$$=\frac{\pi}{4}\left[\frac{1}{\sqrt{3}}\cdot\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)-\frac{1}{2}\ln\left|\frac{x^2-x+1}{x^2+x+1}\right|\right]_0^{\infty}+0-\frac{1}{2}\int_0^{\infty}\frac{1}{1+x^2}\cdot\ln\left|\frac{x^2-x+1}{x^2+x+1}\right|dx$$

$$=\frac{\pi^2}{4\sqrt{3}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1-\sin x\cos x}{1+\sin x\cos x}\right)dx$$

$$=\frac{\pi^2}{4\sqrt{3}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{2-\sin 2x}{2+\sin 2x}\right)dx=\frac{\pi^2}{4\sqrt{3}}-\frac{1}{4}\int_0^{\pi}\ln\left(\frac{2-\sin x}{2+\sin x}\right)dx$$

$$=\frac{\pi^2}{4\sqrt{3}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{2-\sin x}{2+\sin x}\right)dx=\frac{\pi^2}{4\sqrt{3}}+\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{2+\cos x}{2-\cos x}\right)dx$$

$$I=\frac{\pi^2}{8\sqrt{3}}+\frac{1}{4}\left[\frac{8G}{3}-\frac{\pi}{3}\ln\left(2+\sqrt{3}\right)\right]$$

$$=\frac{\pi^2}{8\sqrt{3}}+\frac{2G}{3}-\frac{\pi}{12}\ln\left(2+\sqrt{3}\right)=0.978142302$$

$$\int_0^{\frac{\pi}{2}}\ln\left(\frac{2+\cos x}{2-\cos x}\right)dx=\int_0^{\frac{\pi}{2}}\ln\left(\frac{1+\frac{1}{2}\cos x}{1-\frac{1}{2}\cos x}\right)dx$$

$$=2\int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{2}}\frac{\cos x}{1-y^2\cos^2 x}dy\,dx=2\int_0^{\frac{1}{2}}\frac{1}{y}\int_0^{\frac{\pi}{2}}\frac{d(y\sin x)}{\left(\sqrt{1-y^2}\right)^2+(y\sin x)^2}dy$$

$$=2\int_0^{\frac{1}{2}}\frac{1}{y\sqrt{1-y^2}}\cdot\tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)dy=2\int_0^{\frac{\pi}{6}}\frac{y}{\sin y\cos y}\cos y\,dy=2\int_0^{\frac{\pi}{6}}y\csc y\,dy$$

$$=2\left[y\ln\tan\frac{y}{2}\right]_0^{\frac{\pi}{6}}-2\int_0^{\frac{\pi}{6}}\ln\tan\frac{y}{2}dy=2\frac{\pi}{6}\cdot\ln\left(\tan\frac{\pi}{12}\right)-4\int_0^{\frac{\pi}{12}}\ln(\tan y)\,dy$$

$$=\frac{\pi}{6}\ln\left(\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}\right)-4\left(-\frac{2G}{3}\right)=\frac{8G}{3}+\frac{\pi}{6}\ln\left[\frac{2-\sqrt{3}}{2+\sqrt{3}}\right]=\frac{8G}{3}-\frac{\pi}{3}\ln\left(2+\sqrt{3}\right)$$

Note:

$$\int_0^{\frac{\pi}{12}} \ln(\tan x)\,dx = -\frac{2G}{3}$$

UP.186. If $x_1 = 2; x_2 = 4; x_3 = 10;$

$$x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0; n \in \mathbb{N}; n \geq 1$$

then find:

$$\Omega = \lim_{n\to\infty}\left(x_n^2\left(3^{\frac{1}{x_n}} - 1\right) - x_n \log 3\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

Because the equation $x^3 - 5x^2 + 7x - 3 = 0$ ***has*** $r_1 = r_2 = 1$ ***and*** $r_3 = 3$ ***roots*** $\Rightarrow$

$\lim_{n\to\infty} x_n = \infty$. ***Let*** $\frac{1}{x_n} = x, x \to 0 \Rightarrow$

$$\Omega = \lim_{x\to 0}\frac{3^x - 1}{x^2} - \frac{1}{x}\ln 3 = \lim_{x\to 0}\frac{3^x - 1 - x\ln 3}{x^2} \overset{L'H}{=}$$

$$= \lim_{x\to 0}\frac{3^x \ln 3 - \ln 3}{2x} = \lim_{x\to 0}\frac{\ln 3\,(3^x - 1)}{2x} = \frac{\ln^2 3}{2}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$x_1 = 2, x_2 = 4, x_3 = 10, x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0 \quad (*)$$

$$(*) \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \Leftrightarrow \left[\begin{matrix}\lambda_{1,2} = 1\\ \lambda_3 = 3\end{matrix}\right.$$

$$\Rightarrow x_n = \alpha + \beta n + \gamma \cdot 3^n (n \in \mathbb{N}, n \geq 1, \alpha, \beta, \gamma \in \mathbb{R})$$

$\therefore$ ***Find:*** α, β, γ

$$x_1 = 2 \Rightarrow \alpha + \beta + 3\gamma = 2 \quad (1)$$

$$x_2 = 4 \Rightarrow \alpha + 2\beta + 9\gamma = 4 \quad (2)$$

$$x_3 = 10 \Rightarrow \alpha + 3\beta + 27\gamma = 10 \quad (3)$$

From (1), (2), (3) we have: $\alpha = 1, \beta = 0, \gamma = \frac{1}{3}$

$$\Rightarrow x_n = 1 + \frac{3^n}{3} = 1 + 3^{n-1} \Rightarrow \lim_{n\to\infty} x_n = +\infty$$

$$\Omega_n = \left[x_n^2\left(3^{\frac{1}{x_n}} - 1\right) - x_n \log 3\right] = \frac{\left(3^{\frac{1}{x_n}} - 1\right) - \frac{1}{x_n}\log 3}{\left(\frac{1}{x_n}\right)^2}$$

$$\therefore \textit{Let } t = \frac{1}{x_n}\,(x_n \to +\infty \Rightarrow t \to 0)$$

$$\lim_{t\to0}\frac{3^t - 1 - t\log 3}{t^2} \overset{(L')}{=} \lim_{t\to0}\frac{3^t\log 3 - \log 3}{2t}$$

$$= \frac{\log 3}{2}\lim_{t\to0}\frac{3^t - 1}{t} = \frac{\log 3}{2}\cdot\log 3 = \frac{\log^2 3}{2} \Rightarrow \Omega = \lim_{n\to\infty}\Omega_n = \frac{\log^2 3}{2}$$

Solution 3 by Dimitris Kastriotis-Athens-Greece

$$x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0 \quad \textit{(E)}$$

$$x_1 = 2, x_2 = 4, x_3 = 10. \textit{ Let } x_n = p^n, p \neq 0$$

$$\textit{(E)} \Leftrightarrow p^{n+3} - 5p^{n+2} + 7p^{n+1} - 3p^n = 0 \Leftrightarrow p^n(p^3 - 5p^2 + 7p - 3) = 0$$

$$\overset{p\neq0}{\Leftrightarrow} p^3 - 5p^2 + 7p - 3 = 0 \Leftrightarrow (p-3)(p-1)^2 = 0 \left\langle \begin{matrix} p = 3 \\ p = 1 \end{matrix}\right.$$

$$x_n = c_1 + c_2\cdot n + c_3\cdot 3^n, c_1, c_2, c_3 \in \mathbb{R}, n \geq 1, n \in \mathbb{N}$$

$$\textit{For } n = 1: c_1 + c_2 + 3c_3 = 2$$

$$n = 2: c_1 + 9c_2 + 9c_3 = 4; n = 3: c_1 + 3c_2 + 27c_3 = 10$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 9 \\ 1 & 3 & 27 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 & \vdots & 2 \\ 1 & 2 & 9 & \vdots & 4 \\ 1 & 3 & 27 & \vdots & 10 \end{pmatrix} \overset{\substack{r_2\leftarrow r_2-r_1 \\ r_3\leftarrow r_3-r_1}}{\sim} \begin{pmatrix} 1 & 1 & 3 & \vdots & 2 \\ 0 & 1 & 6 & \vdots & 2 \\ 0 & 2 & 24 & \vdots & 8 \end{pmatrix}$$

$$\overset{r_3\leftarrow r_3-2r_2}{\sim} \begin{pmatrix} 1 & 1 & 3 & \vdots & 2 \\ 0 & 1 & 6 & \vdots & 2 \\ 0 & 0 & 12 & \vdots & 4 \end{pmatrix}$$

$$12c_3 = 4 \Rightarrow c_3 = \frac{1}{3}$$

$$c_q + 6c_3 = q \Rightarrow c_q + 6\cdot\frac{1}{3} = q \Rightarrow c_q = 0$$

$$c_1 + c_q + 3c_3 = q \Rightarrow c_1 + 0 + 1 = q \Rightarrow c_1 = 1$$

$$x_n = 1 + \frac{1}{3}\cdot 3^n = 1 + 3^{n-1}, n = 1, q, \dots$$

$$\Omega = \lim_{n\to\infty}\left[x_n^2\left(3^{\frac{1}{x_n}} - 1\right) - x_n \cdot \log 3\right]$$

$$= \lim_{n\to\infty}\left[(1 + 3^{n-1})^2 \cdot \left(3^{\frac{1}{1+3^{n-1}}} - 1\right) - (1 + 3^{n-1})\log 3\right]$$

$$\overset{k=1+3^{n-1}}{=} \lim_{k\to\infty}\left[k^2\left(3^{\frac{1}{k}} - 1\right) - k\cdot\log 3\right] = \lim_{k\to\infty}\frac{3^{\frac{1}{k}} - 1 - \frac{\log 3}{k}}{\frac{1}{k^2}}$$

$$\overset{x=\frac{1}{k}}{=} \lim_{x\to 0^+}\frac{3^x - 1 - x\log 3}{x^2} \underset{(*)}{\overset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to 0^+}\frac{3^x\cdot\log 3 - \log 3}{2x} \underset{(*)}{\overset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to 0^+}\frac{3^x\cdot\log^2(3)}{2} = \frac{\log^2 3}{2}$$

(*): L'Hôspital's Rule

Solution 4 by Remus Florin Stanca-Romania

If $x_1 = 2; x_2 = 4; x_3 = 10; x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0; n \in \mathbb{N}; n \ge 1$ ***then find:***

$$\Omega = \lim_{n\to\infty}\left(x_n^2\left(3^{\frac{1}{x_n}} - 1\right) - x_n \ln 3\right)$$

$$x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0 \succ x_{n+3} - x_{n+2} - 4x_{n+2} + 4x_{n+1} + 3x_{n+1} - 3x_n = 0 \succ$$

$$\succ (x_{n+3} - x_{n+2}) - 4(x_{n+2} - x_{n+1}) + 3(x_{n+1} - x_n) = 0 \quad \textbf{(1)};$$

Let $a_n = x_n - x_{n-1}$ ***(2);*** $n \ge 2$

$$\overset{(1);(2)}{\succ} a_{n+3} - 4a_{n+2} + 3a_{n+1} = 0 \succ a_{n+3} = 4a_{n+2} - 3a_{n+1}$$

$$t^2 - 4t + 3 = 0 \Rightarrow t_{1,2} = \begin{Bmatrix} t_1 = 3 \\ t_2 = 1 \end{Bmatrix} \Rightarrow a_n = \alpha\cdot 3^n + \beta$$

$$a_2 = 2; a_3 = 6$$

$$a_2 = 9\alpha + \beta = 2$$

$$a_3 = 27\alpha + \beta$$

……………………. "_"

$$\Rightarrow 18\alpha = 4 \Rightarrow \alpha = \frac{2}{9} \Rightarrow \beta = 0 \Rightarrow a_n = 2\cdot 3^{n-2} \Rightarrow x_n - x_{n-1} = 2\cdot 3^{n-2}$$

$$x_2 - x_1 = 2\cdot 3^0$$

$$x_3 - x_2 = 2\cdot 3^1$$

…………………….

$$x_n - x_{n-1} = 2\cdot 3^{n+2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots \text{ “+”}$$

$$\Rightarrow x_n - x_1 = 2\cdot\frac{3^{n-1}-1}{2} \Rightarrow x_n = 3^{n-1}+1$$

$$\Rightarrow \Omega = \lim_{n\to\infty}(3^{n-1}+1)^2\left(3^{\frac{1}{3^{n-1}+1}}-1\right) - (3^{n-1}+1)\ln 3$$

$$\lim_{n\to\infty}(3^{x-1}+1)^2\left(3^{\frac{1}{3^{x-1}+1}}-1\right) - (3^{x-1}+1)\ln 3 =$$

$$= \lim_{n\to\infty}\frac{(3^{x-1}+1)\left(3^{\frac{1}{3^{x-1}+1}}-1\right)-\ln 3}{\frac{1}{3^{x-1}+1}} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{n\to\infty}\frac{\frac{\frac{1}{3^{3^{x-1}+1}}}{3^{x-1}+1}\ln 3 + 1 - 3^{\frac{1}{3^{x-1}+1}}}{\frac{1}{(3^{x-1}+1)^2}}$$

Let $y = \frac{1}{3^{x-1}+1} \Rightarrow \Omega = \lim_{y\to 0}\frac{3^y\ln 3 + 1 - 3^y}{y^2} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{y\to 0}\frac{3^y\ln^2 3y}{2y} = \frac{\ln^2 3}{2} \Rightarrow \Omega = \frac{\ln^2 3}{2}$

Solution 5 by Srinivasa Raghava-AIRMC-India

$7x(n+1) - x(n+2) + x(n+3) = 3x(n)$. *This can be written as*

$3x(n-1) - 2 = 4x(n-1) - 3x(n-2)$. *Comparing to the Lucas Sequence*

$$x(n) = Px(n-1) - Qx(n-2)$$

And by the Inverse binomial transforms yields the generating function

$\frac{1}{3}\left(\frac{1}{1-3x}+\frac{3}{1-x}\right)$ *from this we can see that* $x(n) = \frac{1}{3}(3+3^n) = 1 + 3^{-1+n}$ *and*

$$\lim_{n\to\infty}\left(x(n)^2\left(3^{\frac{1}{x(n)}}-1\right) - \log(3)\,x(n)\right) =$$

$$= \lim_{n\to\infty}\left(-3^{n-1}\log(3) - \log(3) + 3^{\frac{3}{3+3^n}} - 2\times 3^{n-1} + 2\times 3^{n+\frac{3}{3+3^n}-1} - 3^{2n-2} + 3^{2n+\frac{3}{3+3^n}-2} - 1\right)$$

collecting $\log(3)$ *terms and cancellation we have*

$$\lim_{n\to\infty}\left(x(n)^2\left(3^{\frac{1}{x(n)}}-1\right) - x(n)\log(3)\right) = \frac{\log^2(3)}{2}$$

Solution 6 by Tobi Joshua-Nigeria

Given $x_{n+3} - 5x_{n+2} + 7x_{n+1} - 3x_n = 0$

Forming a cubical equation with $x_n = a + bn + c3^n$ *(1)*

$(a, b, c \in \mathbb{R})$ *and* $(n \in \mathbb{N})$

$$u^3-5u^2+7u-3=0, u=1,1,3$$

$$x_0=\frac{4}{3}, x_2=2, \dots$$

From (1) $x_n=a+bn+c3^n$ **then** $x_0=a+c=\frac{4}{3}$

$$a+c=\frac{4}{3} \quad (1)$$

$$x_1=2, a+b+3c=2 \quad (2);\ x_2=4, a+2b+9c=4 \quad (3)$$

$$a_3=10, a+3b+27c=10 \quad (4)$$

Solving (1) – (4), $a=1, b=0, c=\frac{1}{3}$

$$u_n=x_n=1+\frac{3^n}{3}=1+3^{n-1} \quad (2)$$

$$\Omega=\lim_{n\to\infty} x_n^2\left(3^{\frac{1}{x_n}}-1\right)-x_n\ln 3$$

$$\Omega=\lim_{n\to\infty}\frac{\left(3^{\frac{1}{x_n}}-1\right)-\frac{\ln 3}{x_n}}{\left(\frac{1}{x_n}\right)^2}\Rightarrow\lim_{n\to\infty}\frac{\left(3^{\frac{1}{1+3^{n-1}}}\right)-\frac{\ln 9}{1+3^{n-1}}}{\left(\frac{1}{3^{n+1}+1}\right)^2}$$

$\Omega\Rightarrow\lim_{n\to 0}\frac{3^n-1-n\ln 3}{n^2}$. **Using L'Hospital's Rule:**

$$\Omega=\frac{\ln 3}{2}\lim_{n\to 0}\frac{3^n-1}{n}=\frac{\ln 3}{2}(\ln 3)=\frac{\ln^2 3}{2};\ \Omega=\frac{\ln^2 3}{2}$$

UP.187. Find:

$$\Omega=\lim_{n\to\infty}\frac{\left(\sqrt[n+1]{(n+1)!}\right)^4-\left(\sqrt[n]{n!}\right)^4}{n^2\left(\left(\sqrt[n+1]{(2n+1)!!}\right)^2-\left(\sqrt[n]{(2n-1)!!}\right)^2\right)}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{n!^4}\left(\frac{\sqrt[n+1]{(n+1)!^4}}{\sqrt[n]{n!^4}}-1\right)}{n^4\cdot\frac{\sqrt[n]{(2n-1)!!^2}}{n^2}\left(\frac{\sqrt[n+1]{(2n+1)!!^2}}{\sqrt[n]{(2n-1)!!^2}}-1\right)} \quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!^4}}{n^4}=\left(\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^4}}\right)^4=\left(\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)^4=$$

$$=\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\right)^4=\frac{1}{e^4}\quad (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!^2}}{n^2}=\left(\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\right)^2=\lim_{n\to\infty}\left(\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}\right)^2$$

$$=\left(\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n\right)^2=\frac{4}{e^2}\quad (3)$$

Let $x_n=\frac{\sqrt[n+1]{(n+1)!^4}}{\sqrt[n]{n!^4}}$ *and* $y_n=\frac{\sqrt[n+1]{(2n+1)!!^2}}{\sqrt[n]{(2n-1)!!^2}}$

$$\lim_{n\to\infty}\frac{x_n-1}{y_n-1}=\lim_{n\to\infty}\frac{n(x_n-1)}{n(y_n-1)}\quad (3)$$

$$\lim_{n\to\infty}n(x_n-1)=\lim_{n\to\infty}\frac{n\left(e^{\ln x_n}-1\right)}{\ln x_n}x_n=\lim_{n\to\infty}n\ln x_n=$$

$$=\lim_{n\to\infty}\ln x_n^n=\ln\lim_{n\to\infty}x_n^n=\ln\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!^4}}{\sqrt[n]{n!^4}}\right)^n=$$

$$=4\ln\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!^n}}{n!}=4\ln\lim_{n\to\infty}\frac{(n+1)!}{\sqrt[n+1]{(n+1)!}\,n!}=4\ln\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}$$

$$=4\ln\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=4\ln\lim_{n\to\infty}\sqrt[n]{\frac{n^4}{n!}}=4\ln\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=4\ln e=4\quad (4)$$

$$\lim_{n\to\infty}n(y_n-1)=\lim_{n\to\infty}\frac{n\left(e^{\ln y_n}-1\right)}{\ln y_n}\cdot\ln y_n=\lim_{n\to\infty}n\ln y_n=$$

$$=\lim_{n\to\infty}y_n^n=\ln\lim_{n\to\infty}y_n^n=\ln\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(2n+1)!!^2}}{\sqrt[n]{(2n-1)!!^2}}\right)^n=$$

$$=2\ln\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)^4!!}}{(2n-1)!!}=2\ln\lim_{n\to\infty}\frac{(2n+1)!!}{(2n-1)!!\cdot\sqrt[n+1]{(2n+1)!!}}=2\ln\lim_{n\to\infty}\frac{2n+1}{\sqrt[n+1]{(2n+1)!!}}$$

$$=2\ln\lim_{n\to\infty}\frac{2n-1}{\sqrt[n]{(2n-1)!!}}=2\ln\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)^n}{(2n-1)!!}}=2\ln\lim_{n\to\infty}\frac{(2n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{(2n-1)^n}$$

$$=2\ln e=2\quad (5)$$

From (1)+(2) +(3) +(4) +(5)$\Rightarrow \Omega = \frac{1}{2e^2}$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\Omega = \lim_{n\to\infty} \frac{\left(\sqrt[n+1]{(n+1)!}\right)^4 - \left(\sqrt[n]{n!}\right)^4}{n^2\left(\left(\sqrt[n+1]{(2n+1)!!}\right)^2 - \left(\sqrt[n]{(2n-1)!!}\right)^2\right)} =$$

$$= \frac{\lim_{n\to\infty}\left(n^{-3}\left((n+1)^4\left(\sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}\right)^4 - n^4\left(\sqrt[n]{\frac{n!}{n^n}}\right)^4\right)\right)}{\lim_{n\to\infty}\left(n^{-1}\left((n+1)^2\left(\sqrt[n+1]{\frac{(2n+1)!!}{(n+1)^{n+1}}}\right)^2 - n^2\left(\sqrt[n]{\frac{(2n-1)!!}{n^n}}\right)^2\right)\right)}$$

$$= \frac{4\times\lim_{n\to\infty}\left(\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right)^4}{2\times\lim_{n\to\infty}\left(\frac{\frac{(2n+1)!!}{(n+1)^{n+1}}}{\frac{(2n-1)!!}{n^2}}\right)^2} = \frac{2\times\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-4n}}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-2n}\left(\frac{2n+1}{n+1}\right)^2}$$

$$\therefore \Omega = \lim_{n\to\infty} \frac{\left(\sqrt[n+1]{(n+1)!}\right)^4 - \left(\sqrt[n]{n!}\right)^4}{n^2\left(\left(\sqrt[n+1]{(2n+1)!!}\right)^2 - \left(\sqrt[n]{(2n-1)!!}\right)^2\right)} = \frac{1}{2e^2}$$

UP.188. If $m,p > 0; m \geq p; m,n$ – fixed then find in terms of m,p:

$$\Omega = \lim_{n\to\infty} \frac{\left(\sqrt[n+1]{(2n+1)!!}\right)^m - \left(\sqrt[n]{(2n-1)!!}\right)^m}{n^{m-p}\left(\left(\sqrt[n+1]{(n+1)!}\right)^p - \left(\sqrt[n]{n!}\right)^p\right)}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}^m\left[\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m - 1\right]}{n^m \sqrt[n]{n!}^p\left[\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p - 1\right]} \quad (1)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}^m}{n^m} = \left(\lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}}\right)^m \overset{C.D.}{=} \left(\lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!}\right)^m =$$

$$= \left(\lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n\right)^m = \frac{2^m}{e^m} \quad (2)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}^p}{n^p} = \left(\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n}}\right)^p \overset{C.D.}{=} \left(\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right)^p =$$

$$= \left(\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n\right)^p = \frac{1}{e^p} \quad (3)$$

Let $x_n = \left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m$ *and* $y_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p$

$$\lim_{n\to\infty} \frac{x_n - 1}{y_n - 1} = \lim_{n\to\infty} \frac{n(x_n - 1)}{n(y_n - 1)} \quad (4)$$

$$\lim_{n\to\infty} n(x_n - 1) = \lim_{n\to\infty} \frac{n(e^{\ln x_n} - 1)}{\ln x_n} \cdot \ln x_n = \lim_{n\to\infty} n \ln x_n$$

$$= \lim_{n\to\infty} \ln x_n^n = \ln\left(\lim_{n\to\infty} x_n^n\right) = \ln\left(\lim_{n\to\infty}\left(\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m\right)^n\right) =$$

$$= m \ln\left(\lim_{n\to\infty} \frac{\sqrt[n+1]{(2n+1)!!^n}}{(2n-1)!!}\right) = m \ln\left(\lim_{n\to\infty} \frac{(2n+1)!!}{(2n-1)!! \cdot \sqrt[n+1]{(2n+1)!!}}\right) =$$

$$= m \ln\left(\lim_{n\to\infty} \frac{2n+1}{\sqrt[n+1]{(2n+1)!!}}\right) = m \ln\left(\lim_{n\to\infty} \frac{2n-1}{\sqrt[n]{(2n-1)!!}}\right) = m \ln\left(\lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)^n}{(2n-1)!!}}\right)$$

$$= m \ln\left(\lim_{n\to\infty} \frac{(2n+1)^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{(2n-1)^n}\right) = m \ln e = m \quad (5)$$

$$\lim_{n\to\infty} n(y_n - 1) = \lim_{n\to\infty} \frac{n(e^{\ln y_n} - 1)}{\ln y_n} \cdot \ln y_n = \lim_{n\to\infty} n \ln y_n =$$

$$= \lim_{n\to\infty} \ln y_n^n = \ln\left(\lim_{n\to\infty} y_n^n\right) = \ln\left(\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^p\right)^n =$$

$$= p\ln\left(\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n!}\right) = p\ln\left(\lim_{n\to\infty}\frac{(n+1)!}{\sqrt[n+1]{(n+1)!}\cdot n!}\right) = p\ln\left(\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)$$

$$= p\ln\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right) = p\ln e = p \quad (6)$$

$$\text{From } (1)+(2)+(3)+(4)+(5)+(6) \Rightarrow \Omega = \frac{\frac{2^m}{e^m}}{\frac{1}{e^p}}\cdot\frac{m}{p} = \frac{m}{p}\cdot 2^m\cdot e^{p-m}$$

UP.189. Find:

$$\Omega = \lim_{n\to\infty}\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{m+1} - \left(\sqrt[n]{n!}\right)^{m+1}}{\left(\sqrt[n]{(2n-1)!!}\right)^{m}};\ m\in\mathbb{N}, m - \text{fixed.}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$$\Omega = \lim_{n\to\infty}\frac{\sqrt[n]{n!}^{m+1}\cdot\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1} - 1\right)}{\sqrt[n]{(2n-1)!!}^{m}} =$$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{n!}^{m}}{\sqrt[n]{(2n-1)!!}^{m}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot n\left(\left(\frac{\sqrt[n]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1} - 1\right) \quad (1)$$

$$\lim_{n\to\infty}\left(\sqrt[n]{\frac{n!}{(2n-1)!!}}\right)^m = \left(\lim_{n\to\infty}\sqrt[n]{\frac{n!}{(2n-1)!!}}\right)^m \overset{CD}{=} \left(\lim_{n\to\infty}\frac{(n+1)}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n!}\right)^m$$

$$= \left(\lim_{n\to\infty}\frac{n+1}{2n+1}\right)^m = \frac{1}{2^m} \quad (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{CD}{=} \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} =$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e} \quad (3)$$

$$\text{Let } x_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}$$

$$\lim_{n\to\infty} n(x_n - 1) = \lim_{n\to\infty}\frac{n\left(e^{\ln x_n} - 1\right)}{\ln x_n}\cdot\ln x_n =$$

$$\lim_{n\to\infty} n\ln x_n = \lim_{n\to\infty} \ln x_n^n = \ln\left(\lim_{n\to\infty} x_n^n\right) =$$

$$= \ln\left(\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}\right)^n = (m+1)\ln\left(\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)!}}{n!}\right) =$$

$$= (m+1)\ln\left(\lim_{n\to\infty}\frac{(n+1)!}{\sqrt[n+1]{(n+1)!}\,n!}\right) = (m+1)\ln\left(\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)$$

$$= (m+1)\ln\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right) = (m+1)\ln e = m+1 \quad \textit{(4)}$$

From (1)+(2)+(3)+(4)$\Rightarrow \Omega = \frac{1}{2^m}\cdot\frac{1}{e}(m+1) = \frac{m+1}{e\cdot 2^m}$

UP.190. If $a_n, b_n > 0; n \geq 1; \lim_{n\to\infty}\frac{a_{n+1}}{na_n} = a; \lim_{n\to\infty}\frac{b_{n+1}}{n^3 b_n} = b; a, b > 0$

then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[2n+2]{a_{n+1}\cdot b_{n+1}}} - \frac{n^3}{\sqrt[2n]{a_n\cdot b_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu; Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty}\frac{n^2}{\sqrt[2n]{a_n b_n}}\cdot n\left(\frac{(n+1)^3}{n^3}\cdot\frac{\sqrt[2n]{a_n b_n}}{\sqrt[2n+2]{a_{n+1}b_{n+1}}} - 1\right) \quad \textit{(1)}$$

$$\lim_{n\to\infty}\frac{n^2}{\sqrt[2n]{a_n b_n}} = \lim_{n\to\infty}\sqrt[2n]{\frac{n^{4n}}{a_n b_n}} = \sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{n^{4n}}{a_n b_n}}} \overset{C.D.}{=}$$

$$= \sqrt{\lim_{n\to\infty}\frac{(n+1)^{4n+4}}{a_{n+1}b_{n+1}}\cdot\frac{a_n b_n}{n^{4n}}} = \sqrt{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{4n}\cdot\frac{na_n}{a_{n+1}}\cdot\frac{n^3 b_n}{b_n+1}\cdot\frac{(n+1)^4}{n^3}}$$

$$= \sqrt{e^4\cdot\frac{1}{a}\cdot\frac{1}{b}} = \frac{e^2}{\sqrt{ab}} \quad \textit{(2)}$$

$$\lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^3\frac{\sqrt[2n]{a_n b_n}}{\sqrt[2n+2]{a_{n+1}b_{n+1}}} - 1\right) = \lim_{n\to\infty} n[x_n - 1] =$$

Let $x_n = \left(\frac{n+1}{n}\right)^3\frac{\sqrt[2n]{a_n b_n}}{\sqrt[2n+2]{a_{n+1}b_{n+1}}}$

$$= \lim_{n\to\infty}\frac{n\left(e^{\ln x_n}-1\right)}{\ln x_n}\cdot \ln x_n = \lim_{n\to\infty} n\ln x_n = \lim_{n\to\infty}\ln x_n^n =$$

$$= \ln\left(\lim_{n\to\infty} x_n^n\right) = \ln\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{3n}\left(\frac{\sqrt[2n]{a_nb_n}}{\sqrt[2n+2]{a_{n+1}b_{n+1}}}\right)^n\right) =$$

$$\ln e^3 \lim_{n\to\infty}\sqrt{\frac{\left(\sqrt[n]{a_nb_n}\right)^n}{\sqrt[n+1]{(a_{n+1}b_{n+1})^n}}} = \ln e^3\sqrt{\lim_{n\to\infty}\frac{a_nb_n\cdot\sqrt[n+1]{a_{n+1}b_{n+1}}}{a_{n+1}b_{n+1}}}$$

$$= \ln e^3\sqrt{\lim_{n\to\infty}\frac{na_nn^3b_n}{a_{n+1}b_{n+1}}\cdot\frac{\sqrt[n]{a_nb_n}}{n^4}} =$$

$$\ln e^3\sqrt{\frac{1}{ab}\lim_{n\to\infty}\sqrt[n]{\frac{a_nb_n}{n^{4n}}}} = \ln e^3\sqrt{\frac{1}{ab}\lim_{n\to\infty}\frac{a_{n+1}b_{n+1}}{(n+1)^{4n+4}}\cdot\frac{n^{4n}}{a_nb_n}}$$

$$= \ln e^3\sqrt{\frac{1}{ab}\lim_{n\to\infty}\frac{a_{n+1}}{na_n}\cdot\frac{b_{n+1}}{n^3b_n}\cdot\frac{n^4}{(n+1)^4}\left(\frac{n}{n+1}\right)^{4n}} =$$

$$= \ln e^3\sqrt{\frac{1}{ab}\cdot ab\cdot\frac{1}{e^2}} = \ln\frac{e^3}{e} = \ln e^2 = 2 \quad \textit{(3)}$$

From (1)+(2)+(3)$\Rightarrow \Omega = \frac{2e^2}{\sqrt{ab}}$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\frac{n^3}{\sqrt[2n]{a_nb_n}}\left(\left(\frac{n+1}{n}\right)^3\cdot\frac{\sqrt[2n]{a_nb_n}}{\sqrt[2n+2]{a_{n+1}b_{n+1}}}-1\right) \quad \textit{(a)}$$

$$\lim_{n\to\infty}\frac{n^2}{\sqrt[2n]{a_nb_n}} = \lim_{n\to\infty} e^{\ln\left(\frac{n^2}{\sqrt[2n]{a_nb_n}}\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(\frac{n^{4n}}{a_nb_n}\right)}{2n}} \overset{Stolz\ Cesaro}{=}$$

$$= \lim_{n\to\infty} e^{\frac{\ln\left(\frac{(n+1)^{4n+4}}{n^{4n}}\cdot\frac{a_nb_n}{a_{n+1}b_{n+1}}\right)}{2}} = \lim_{n\to\infty}\frac{(n+1)^{2n+2}}{n^{2n}}\cdot\sqrt{\frac{a_nb_n}{a_{n+1}b_{n+1}}} =$$

$$= e^2\lim_{n\to\infty}\sqrt{\frac{(n+1)^4a_nb_n}{a_{n+1}b_{n+1}}} = e^2\lim_{n\to\infty}\sqrt{\frac{n^4}{(n+1)^4}\cdot(n+1)^4\cdot\frac{a_nb_n}{a_{n+1}b_{n+1}}} = e^2$$

$$\lim_{n\to\infty}\sqrt{\frac{na_n}{a_{n+1}}\cdot\frac{n^3b_n}{b_{n+1}}} = \frac{e^2}{\sqrt{ab}} \Rightarrow \lim_{n\to\infty}\frac{(n+1)^2}{\sqrt[2n+2]{a_{n+1}b_{n+1}}} = \frac{e^2}{\sqrt{ab}} \quad \textit{(1)}$$

$$\lim_{n\to\infty}\frac{n^2}{\sqrt[2n]{a_nb_n}}=\frac{e^2}{\sqrt{ab}}\quad (2)$$

$$----------\ \text{“:”}$$

$$\overset{(1);(2)}{\Rightarrow}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2\cdot\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}=1$$

$$\overset{(a)}{\Rightarrow}\Omega=\lim_{n\to\infty}\frac{e^2}{\sqrt{ab}}\cdot n\cdot$$

$$\cdot\left(\left(\frac{n+1}{n}\right)^3\cdot\frac{\sqrt[2n]{a_nb_n}}{\sqrt[2n+2]{a_{n+1}b_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^2\cdot\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}-\left(\frac{n}{n+1}\right)^2\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}\right)=$$

$$=\frac{e^2}{\sqrt{ab}}\lim_{n\to\infty}n\left(\frac{n+1}{n}-\left(\frac{n}{n+1}\right)^2\cdot\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}\right)\quad (3)$$

$$\lim_{n\to\infty}n\left(\frac{n+1}{n}-\left(\frac{n}{n+1}\right)^2\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}\right)=1-\lim_{n\to\infty}n\left(\left(\frac{n}{n+1}\right)^2\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}\right)\quad (4)$$

$$\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^2\cdot\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}-1\right)=\lim_{n\to\infty}n\left(\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}-\left(\frac{n+1}{n}\right)^2\right)=$$

$$=\lim_{n\to\infty}n\left(\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}-1\right)-\lim_{n\to\infty}n\left(\left(\frac{n+1}{n}\right)^2-1\right)\quad (5)$$

$$l_1=\lim_{n\to\infty}n\left(\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}-1\right)\ \text{and}\ l_2=\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^2-1\right)$$

$$l_1=\lim_{n\to\infty}n\cdot\frac{\left(e^{\ln\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}}-1\right)}{\ln\left(\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}\right)}\cdot\ln\left(\frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_nb_n}}\right)$$

$$=\lim_{n\to\infty}n\ln\left(\frac{(a_{n+1}b_{n+1})^{\frac{1}{2n+2}}}{(a_nb_n)^{\frac{1}{2n}}}\right)=\lim_{n\to\infty}(n+1)\ln\left(\frac{(a_{n+1}b_{n+1})^{\frac{1}{2n+2}}}{(a_nb_n)^{\frac{1}{2n}}}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{2}\ln\left(\frac{a_{n+1}b_{n+1}}{a_nb_n}\cdot\frac{1}{\sqrt[n]{a_nb_n}}\right)=\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{a_{n+1}}{na_n}\cdot\frac{b_{n+1}}{n^3b_n}\cdot\frac{n^4}{\sqrt[n]{a_nb_n}}\right)=$$

$$=\frac{1}{2}\left(\ln(ab)+\lim_{n\to\infty}\ln\left(\frac{n^4}{\sqrt[n]{a_nb_n}}\right)\right)=\ln(\sqrt{ab})+\ln\left(\frac{e^2}{\sqrt{ab}}\right)=\ln(e^2)=2$$

$$\Rightarrow l_1=2$$

$$l_2 = \lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^2 - 1\right) = \lim_{n\to\infty} n\left(\frac{n+1}{n} - 1\right)\left(\frac{n+1}{n} + 1\right) =$$

$$= 2 \overset{(5)}{\Rightarrow} \lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^2 \cdot \frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_n b_n}} - 1\right) = l_1 - l_2 =$$

$$= 0 \overset{(4)}{\Rightarrow} \lim_{n\to\infty} n\left(\frac{n+1}{n} - \left(\frac{n}{n+1}\right)^2 \frac{\sqrt[2n+2]{a_{n+1}b_{n+1}}}{\sqrt[2n]{a_n b_n}}\right) =$$

$$= 1 - 0 = 1 \overset{(3)}{\Rightarrow} \Omega = \frac{e^2}{\sqrt{ab}} \cdot 1 \Rightarrow \Omega = \frac{e^2}{\sqrt{ab}}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a$ ***and*** $\lim_{n\to\infty} \frac{b_{n+1}}{n^3 b_n} = b$ ***then***

$$\lim_{n\to\infty} \frac{\sqrt[2n]{a_n \cdot b_n}}{n^2} = \sqrt{\lim_{n\to\infty} \sqrt[n]{\frac{a_n \cdot b_n}{n^{4n}}}} \overset{CAESARO\ STOLZ}{=} \sqrt{\lim_{n\to\infty}\left(\frac{b_{n+1}}{n^3 b_n} \cdot \frac{a_{n+1}}{na_n} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^4} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^{4n}}\right)}$$

$= \frac{\sqrt{ab}}{e^2}$, ***let*** $u_n = \frac{\sqrt[2n]{a_n \cdot b_n}}{n^3} \cdot \frac{(n+1)^3}{\sqrt[2n+2]{a_{n+1} \cdot b_{n+1}}}$ ***for all*** $n \in \mathbb{N}$

$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt[2n]{a_n b_n}}{n^2} \cdot \frac{(n+1)^2}{\sqrt[2n+2]{a_{n+1} b_{n+1}}}\left(\frac{n}{1+n}\right)^2 = 1$ ***hence*** $\frac{u_n - 1}{\ln u_n} \to 1$

$$\lim_{n\to\infty} u_n^n = \sqrt{\lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^4} \cdot \left(1+\frac{1}{n}\right)^{6n} \cdot \frac{1}{\frac{a_{n+1}}{na_n}} \cdot \frac{1}{\frac{b_{n+1}}{n^3 b_n}} \cdot \left(\frac{n}{n+1}\right)^4} = e$$

$$\therefore \lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[2n+2]{a_{n+1}b_{n+1}}} - \frac{n^3}{\sqrt[2n]{a_n b_n}}\right) = \lim_{n\to\infty}\left(\frac{n^2}{\sqrt[2n]{a_n b_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n\right) = \frac{e^2}{\sqrt{ab}}$$

(Answer)

UP.191. Let be: $\omega = \sum_{n=1}^{\infty} \frac{1}{\left[\sqrt[3]{(n^3+2n+1)}\right]^2}$; $[*]$ **- great integer function. Find:**

$$\Omega = \lim_{n\to\infty} n\left(\omega - \sum_{k=1}^{n} \frac{1}{k^2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Zaharia Burghelea-Romania

We have that for $n \geq 1$:

$$n^3 < n^3 + 2n + 1 < n^3 + 2n + 1 + 3n^2 + n = (n+1)^3$$

$$\Rightarrow n < \sqrt[3]{n^3 + 2n + 1} < n + 1 \Rightarrow \left[\sqrt[3]{n^3 + 2n + 1}\right] = n$$

Using the above relation yields:

$$\omega = \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \Omega = \lim_{n\to\infty} n\left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{k=1}^{n} \frac{1}{k^2}\right) =$$

$$= \lim_{n\to\infty} n \sum_{j=n+1}^{\infty} \frac{1}{j^2} = \lim_{n\to\infty} n \sum_{j=0}^{\infty} \frac{1}{(j+n+1)^2} = \lim_{n\to\infty} n \cdot \psi_1(n+1)$$

Where the trigamma function $\psi_1(x)$ can be asymptotically approximated as:

$$\psi_1(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + O\left(\frac{1}{x^6}\right)$$

It follows that:

$$\Omega = \lim_{n\to\infty} n\left(\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{6(n+1)^3} + O\left(\frac{1}{n^4}\right)\right) = 1$$

Solution 2 by Marian Ursărescu-Romania

Another approach: Obvious: $\left[\sqrt[3]{n^3 + 2n + 1}\right] = n$, because

$$n < \sqrt[3]{n^3 + 2n + 1} < n + 1 \Rightarrow \omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} \Rightarrow$$

$$\Omega = \lim_{n\to\infty} n\left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}\right) = \lim_{n\to\infty} \frac{\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}}{\frac{1}{n}} =$$

and now, using Cesaró-Stolz lemma for $\frac{0}{0}$

$$= \lim_{n\to\infty} \frac{\frac{\pi^2}{6} - \sum_{k=1}^{n+1} \frac{1}{k^2} - \frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{1}{k^2}}{\frac{1}{n+1} - \frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n+1^2}}{\frac{n-n-1}{n(n+1)}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Solution 3 by Naren Bhandari-Bajura-Nepal

Observe that: $n^3 + 2n + 1 = n^3\left(1 + \frac{2}{n^2} + \frac{1}{n^3}\right)$

Therefore: $\left\lfloor\sqrt[3]{n^3+2n+1}\right\rfloor = \left\lfloor n\left(\sqrt[3]{1+\frac{2}{n^2}+\frac{1}{n^3}}\right)\right\rfloor$

For $n=1$ ***expression above is*** 1 ***and using Fractional Binomial Theorem we have:***

$n\left(1+\frac{1}{3}\left(1+\frac{2}{n^2}+\frac{1}{n^3}\right)+\cdots\right)$ $\forall n\geq 2$ ***and the floor value we have***

$\left\lfloor\sqrt[3]{n^3+2n+1}\right\rfloor = n$. ***Thus***

$$\omega = \sum_{n=1}^{\infty}\frac{1}{\left\lfloor\sqrt[3]{n^3+2n+1}\right\rfloor^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Omega = \lim_{n\to\infty} n\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right) = \frac{1}{\frac{1}{n}}\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)$$

we have the limit $\frac{0}{0}$ ***form so by Stolz – Cesaro Theorem***

$$\Omega = \lim_{n\to\infty}\frac{1}{\frac{1}{n+1}-\frac{1}{n}}\left(\sum_{k=1}^{n+1}\frac{1}{k^2}-\sum_{k=1}^{n}\frac{1}{k^2}\right) = \lim_{n\to\infty}\frac{n(n+1)}{(n+1)^2} = 1$$

UP.192. Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{p=1}^{n}\left(\frac{1}{\sum_{k=1}^{p}\left\{\frac{kp}{p+1}\right\}}\right)-2\log(2n+1)\right)$$

$\{*\} = * - [*]$; $[*]$ **- great integer function.**

Proposed by Daniel Sitaru – Romania

Solution 1 by Yubian Andres Bedoya Henao-Colombia

$$\Omega = \lim_{n\to\infty}\left[\sum_{p=1}^{n}\left(\frac{1}{\sum_{k=1}^{p}\left\{\frac{kp}{p+1}\right\}}\right)-2\ln(2n+1)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{p=1}^{n}\left(\frac{1}{\sum_{k=1}^{p}\left\{k-\frac{k}{p+1}\right\}}\right)-2\ln(2n+1)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{p=1}^{n}\left(\frac{1}{\sum_{k=1}^{p}\left\{1-\frac{k}{p+1}\right\}}\right)-2\ln(2n+1)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{p=1}^{n}\left(\frac{1}{p-\frac{p(p+1)}{2(p+1)}}\right)-2\ln(2n+1)\right] = \lim_{n\to\infty}\left[2\sum_{p=1}^{n}\frac{1}{p}-2\ln(2n+1)\right]$$

$$= 2\lim_{n\to\infty}\left[\sum_{p=1}^{n}\frac{1}{p}-\ln(n)-\ln\left(\frac{2n+1}{n}\right)\right]$$

$$\Omega = 2(\gamma-\ln(2))$$

Solution 2 by Kamel Benaicha-Algeirs-Algerie

$$\Omega = \lim_{n\to\infty}\left(\sum_{p=1}^{n}\frac{1}{\sum_{k=1}^{p}\left\{\frac{kp}{p+1}\right\}}-2\ln(2n+1)\right)$$

$$S(p) = \sum_{k=1}^{p}\left\{\frac{kp}{p+1}\right\} = \sum_{s=0}^{p-1}\left\{\frac{(p-s)(p+1-1)}{p+1}\right\}$$

$$\therefore S(p) = \sum_{s=0}^{p-1}\left\{p-1-s+\frac{s+1}{p+1}\right\} = \frac{1}{p+1}\sum_{s=0}^{p-1}(s+1) = \frac{1}{p+1}\left(\frac{p(p-1)}{2}+p\right) =$$

$$= \frac{p(p+1)}{2(p+1)} = \frac{p}{2}$$

$$\Omega = 2\lim_{n\to\infty}\left(\sum_{p=1}^{n}\frac{1}{p}-\ln(2n+1)\right) = 2\left(\lim_{n\to\infty}\left(\left(\sum_{p=1}^{n}\frac{1}{p}-\ln(n)\right)-\lim_{n\to\infty}\ln\left(\frac{2n+1}{n}\right)\right)\right)$$

$= 2(\gamma-\ln(2))$, ***where γ is Euler – Mascheroni's constant.***

$$\therefore \lim_{n\to\infty}\left(\sum_{p=1}^{n}\frac{1}{\sum_{k=1}^{p}\left\{\frac{kp}{p+1}\right\}}-2\ln(2n+1)\right) = 2(\gamma-\ln(2))$$

UP.193.

$$\omega(n)=n\prod_{i=2}^{n}\left(\frac{i^3+1}{i^3-1}\right)$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\omega^2(n)\left(1+\frac{1}{\omega(n)}\right)^{\frac{1}{\omega(n)}}-\omega^2(n)\cos\left(\frac{1}{\omega^2(n)}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\omega(n)=n\prod_{i=2}^{n}\left(\frac{i^3+1}{i^3-1}\right)=n\cdot\prod_{i=2}^{n}\frac{(i+1)(i^2-i+1)}{(i-1)(i^2+i+1)}=$$

$$=u\cdot\frac{3}{1}\cdot\frac{3}{7}\cdot\frac{4}{2}\cdot\frac{7}{13}\cdot\ldots\cdot\frac{(n+1)}{(n-1)}\cdot\frac{n^2-n+1}{n^2+n+1}=$$

$$=n\cdot\frac{(n+1)!}{(n-1)!}\cdot\frac{3}{n^2+n+1}=n(n+1)n\cdot\frac{3}{n^2+n+1}=\frac{3n^2(n+1)}{n^2+n+1}\to\infty$$

$$\Omega=\lim_{n\to\infty}\omega^2(n)\left[\left(1+\frac{1}{\omega(n)}\right)^{\frac{1}{\omega(n)}}-\cos\frac{1}{\omega^2(n)}\right]=$$

$$\left.\begin{array}{c}\lim\limits_{n\to\infty}\dfrac{\left(1+\frac{1}{\omega(n)}\right)^{\frac{1}{\omega(n)}}-\cos\frac{1}{\omega^2(n)}}{\frac{1}{\omega^2(n)}}\\ Let\,\dfrac{1}{\omega(n)}=x\to 0\end{array}\right\}\Rightarrow$$

$$\Omega=\lim_{x\to0}\frac{(1+x)^x-\cos x^2}{x^2}=\lim_{x\to0}\frac{(1+x)^x-1}{x^2}+\lim_{x\to0}\frac{1-\cos x^2}{x^2}\overset{L'H}{=}$$

$$=\lim_{x\to0}\frac{(1+x)^x\left[\ln(1+x)+\frac{x}{1+x}\right]}{2x}+\lim_{x\to0}\frac{\sin x^2\cdot 2x}{2x}=$$

$$=\lim_{x\to0}(1+x)^x\left[\frac{\ln(1+x)}{2x}+\frac{1}{2(1+x)}\right]+\lim_{x\to0}\sin x^2=\frac{1}{2}+\frac{1}{2}=1$$

Solution 2 by Srinivasa Raghava-AIRMC-India

$$w(n) = n\prod_{i=2}^{n}\frac{i^3+1}{i^3-1} = n\prod_{i=2}^{n}\frac{(i+1)(i^2-i+1)}{(i-1)(i^2+i+1)} =$$

$$= n\prod_{i=2}^{n}\left(-\frac{2(i+2)}{3(i^2+i+1)}+\frac{2}{3(i-1)}+1\right) = \frac{3n^2(n+1)}{2(n^2+n+1)}$$ ***(partial fraction decomposition)***

Now, if we observe the limit

$$\lim_{n\to\infty}\left(w(n)^2\left(1+\frac{1}{w(n)}\right)^{\frac{1}{w(n)}} - w(n)^2\cos\left(\frac{1}{w(n)^2}\right)\right) =$$

$$= \lim_{n\to\infty}\left(\frac{9n^4(n+1)^2\left(\frac{2}{3n^2}+1+\frac{2}{3(n+1)}\right)^{\frac{2}{3}\left(\frac{1}{n^2}+\frac{1}{n+1}\right)}}{4(n^2+n+1)^2} - \frac{9n^4(n+1)^2\cos\left(\frac{4\left(n^2+n+1\right)^2}{9n^4(n+1)^2}\right)}{4(n^2+n+1)^2}\right)$$

By convergence ratio test, we can see that:

$$\lim_{n\to\infty}\cos\left(\frac{1}{w(n^2)}\right) = \lim_{n\to\infty}\cos\left(\frac{4\left(n^2+n+1\right)^2}{9n^4(n+1)^2}\right) = 1 \quad \textit{(A)}$$

$$\lim_{n\to\infty}\left(\frac{2}{3n^2}+\frac{2}{3(n+1)}+1\right)^{\frac{2}{3}\left(\frac{1}{n^2}+\frac{1}{n+1}\right)} = 1 \qquad \textit{(B)}$$

By filtering the common terms and in the view of A & B we obtain:

$$\lim_{n\to\infty}\left(w(n)^2\left(1+\frac{1}{w(n)}\right)^{\frac{1}{w(n)}} - w(n)^2\cos\left(\frac{1}{w(n)^2}\right)\right) = 1$$

Solution 3 by Naren Bhandari-Bajura-Nepal

Given

$$\omega(n) = n\prod_{i=2}^{n}\left(\frac{i^3+1}{i^3-1}\right)$$

$$= n\prod_{i=2}^{n}\left[\frac{(i+1)(i^2-i+1)}{(i-1)(i^2+i+1)}\right] = n\prod_{i=2}^{n}\left[\frac{i+1}{i-1}\right]\prod_{i=2}^{n}\left[\frac{i^2-i+1}{i^2+i+1}\right]$$

$$\therefore \omega(n) = n\left[\frac{n(n+1)}{2}\right]\left[\frac{3}{n^2+n+1}\right] = \frac{3n^2(n+1)}{2(n^2+n+1)}$$

$$\textit{Let } \Omega_1 = \lim_{n\to\infty}\left(\omega^2(n)\left(1+\frac{1}{\omega(n)}\right)^{\frac{1}{\omega(n)}}\right)$$

$$= \lim_{n\to\infty}\exp\left(\frac{2(n^2+n+1)}{n^2(n+1)}\ln\left(\frac{9(n^4(n^2+n+1)}{4(n^2+n+1)^2}+\frac{3\left(n^2(n+1)\right)}{2(n^2+n+1)}\right)\right) = e^0 = 1$$

$$\textit{and } \Omega_2 = \lim_{n\to\infty}\omega^2(n)\cos\left(\frac{1}{\omega^2(n)}\right)$$

$$= \frac{9n^4(n+1)^2}{4(n^2+n+1)^2}\sum_{k=0}^{\infty}\left(\frac{(-1)^k 4(n^2+n+1)^2}{9n^4(n+1)^2(2k)!}\right) = \frac{9n^6}{4n^6}\left(\frac{1+\frac{1}{n}}{\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}\right)^2(1-0) = 0$$

$$\textit{Therefore } \Omega = \Omega_1 + \Omega_2 = 1$$

UP.194. Let a, b be two real numbers with $0 \le a < b$. Calculate the next limit:

$$\lim_{n\to\infty}\int_a^b \sqrt[n]{(-x+a+b)^n+\left(\frac{a+2b}{3}\right)^n+x^n}\,dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by Kamel Benaicha-Algeirs-Algerie

$$\Omega = \lim_{n\to+\infty}\int_a^b \sqrt[n]{(a+b-x)^n+\left(\frac{(a+2b)}{3}\right)^n+x^n}\,dx \;/\; 0 < a \le b$$

$$\textit{Let be } (\alpha,\beta,\lambda) \in {\mathbb{R}_+^*}^3 \textit{ and let be } \theta = \max(\alpha,\beta,\lambda)$$

$$\lim_{n\to\infty}(\alpha^n+\beta^n+\lambda^n)^{\frac{1}{n}} = \theta\lim_{n\to\infty}\left(\left(\frac{\alpha}{\theta}\right)^n+\left(\frac{\beta}{\theta}\right)^n+\left(\frac{\lambda}{\theta}\right)^n\right)^{\frac{1}{n}} = \theta$$

$$\left(\lim_{n\to\infty}\left(\left(\frac{\alpha}{\theta}\right)^n+\left(\frac{\beta}{\theta}\right)^n+\left(\frac{\lambda}{\theta}\right)^n\right) = 1\right). \textit{ Then:}$$

$$\textit{On } \left[a;\frac{2a+b}{3}\right[: \max\left(a+b-x;\frac{a+2b}{3};x\right) = a+b-x,$$

$$\textit{On } \left[\frac{2a+b}{3};\frac{a+2b}{3}\right[: \max\left(a+b-x;\frac{a+2b}{3};x\right) = \frac{a+2b}{3},$$

$$\textit{On } \left[\frac{a+2b}{3};b\right] : \max\left(a+b-x;\frac{a+2b}{3};x\right) = x,$$

$$\therefore \Omega = \int_a^{\frac{b+2a}{3}} (a+b-x)\,dx + \frac{a+2b}{3}\int_{\frac{b+2a}{3}}^{\frac{a+2b}{3}} dx + \int_{\frac{a+2b}{3}}^{b} x\,dx$$

$$= \frac{b^2-a^2}{3} - \frac{1}{2}\left(\left(\frac{b+2a}{3}\right)^2 - a^2\right) + \frac{a+2b}{3}\left(\frac{a+2b}{3} - \frac{b+2a}{3}\right) + \frac{1}{2}\left(b^2 - \left(\frac{a+2b}{3}\right)^2\right)$$

$$= \frac{b^2-a^2}{3} - \frac{(b-a)(5a+b)}{18} + \frac{(b-a)(a+2b)}{9} + \frac{(b-a)(a+5b)}{18}$$

$$= \frac{(b-a)(2a+7b)}{9}$$

$$\therefore \lim_{n\to+\infty}\int_a^b \sqrt[n]{(a+b-x)^n + \left(\frac{(a+2b)}{3}\right)^n + x^n}\,dx = \frac{(b-a)(2a+7b)}{9}$$

UP.195. Calculate the integral:

$$\int_0^\infty \frac{x^2\ln(x+1)}{x^4-x^2+1}dx$$

It is required to express the integral value with the usual mathematical constants, without using values of special functions.

Proposed by Vasile Mircea Popa – Romania

Solution by Zaharia Burghelea-Romania

Denote: $I = \int_0^\infty \frac{x^2\ln(1+x)}{x^4-x^2+1}dx$

We now split the integral from $[0,1]$ ***to*** $[1,\infty)$ ***and in the second integral we substitute***

$x = \frac{1}{x}$ ***in order to arrive at:***

$$I = \int_0^1 \frac{x^2\ln(1+x)}{x^4-x^2+1}dx + \int_0^1 \frac{\ln(1+x)-\ln x}{x^4-x^2+1}dx =$$

$$= \int_0^1 \frac{(x^2+1)\ln(1+x)}{x^4-x^2+1}dx - \int_0^1 \frac{\ln x}{x^4-x^2+1}dx = I_1 - I_2$$

In I_1 ***substituting*** $x = \frac{1-t}{1+t} \Rightarrow dx = -\frac{2}{(1+t)^2}dt$ ***we get:***

$$I_1 = \int_0^1 \frac{(x^2+1)\ln(1+x)}{x^4-x^2+1}dx = 4\int_0^1 \frac{(t^2+1)(\ln 2 - \ln(1+t))}{t^4+14t^2+1}dt$$

$$= \ln 2 \arctan\left(\frac{4t}{1-t^2}\right)\Big|_0^1 - 4\int_0^1 \frac{t^2+1}{t^4+14t^2+1}\ln(1+t)\,dt$$

$$= \frac{\pi}{2}\ln 2 - (2+\sqrt{3})\left(\int_0^1 \frac{\ln(1+t)}{t^2+(2+\sqrt{3})^2}dt + \int_0^1 \frac{\ln(1+t)}{(2+\sqrt{3})^2t^2+1}dt\right)$$

$$\int_0^1 \frac{\ln(1+t)}{t^2+(2+\sqrt{3})^2}dt = \int_0^\infty \frac{\ln(1+t)}{t^2+(2+\sqrt{3})^2}dt + \int_0^1 \frac{\ln t}{(2+\sqrt{3})^2t^2+1}dt$$

$$\Rightarrow I_1 = \frac{\pi}{2}\ln 2 - (2+\sqrt{3})\left(\int_0^\infty \frac{\ln(1+t)}{t^2+(2+\sqrt{3})^2}dt + \int_0^1 \frac{\ln t}{(2+\sqrt{3})^2t^2+1}dt\right)$$

Substituting $t = (2+\sqrt{3})x$ ***respectively*** $(2+\sqrt{3})t = x$ ***we get:***

$$= \frac{\pi}{2}\ln 2 - \left(\int_0^\infty \frac{\ln(1+(2+\sqrt{3})x)}{1+x^2}dx + \int_0^{2+\sqrt{3}} \frac{\ln\left(\frac{x}{2+\sqrt{3}}\right)}{1+x^2}dx\right)$$

$$= \frac{\pi}{2}\ln 2 + \ln(2+\sqrt{3})\arctan(x)\Big|_0^{2+\sqrt{3}} - \int_0^\infty \frac{\ln(1+(2+\sqrt{3})x)}{1+x^2}dx - \int_0^{2+\sqrt{3}} \frac{\ln(x)}{1+x^2}dx$$

$$= \frac{\pi}{2}\ln 2 + \frac{5\pi}{12}\ln(2+\sqrt{3}) - \left(J_1(2+\sqrt{3}) + J_2(2+\sqrt{3})\right)$$

$$J_1(a) = \int_0^\infty \frac{\ln(1+ax)}{1+x^2}dx \Rightarrow J'(a) = \int_0^\infty \frac{x}{(1+x^2)(1+ax)}dx =$$

$$= \frac{a}{1+a^2}\int_0^\infty \frac{1}{1+x^2}dx + \frac{1}{1+a^2}\int_0^\infty \left(\frac{x}{1+x^2} - \frac{a}{1+ax}\right)dx =$$

$$=\frac{\pi}{2}\cdot\frac{a}{1+a^2}+\frac{1}{1+a^2}\ln\left(\frac{\sqrt{1+x^2}}{1+ax}\right)\Big|_0^\infty=\frac{\pi}{2}\cdot\frac{a}{1+a^2}-\frac{\ln a}{1+a^2}$$

$$J_1(0)=0\Rightarrow J_1\left(2+\sqrt{3}\right)=\int_0^{2+\sqrt{3}}\left(\frac{\pi}{2}\cdot\frac{a}{1+a^2}-\frac{\ln a}{1+a^2}\right)da$$

$$J_1\left(2+\sqrt{3}\right)+J_2\left(2+\sqrt{3}\right)=\int_0^{2+\sqrt{3}}\frac{a}{1+a^2}da-\int_0^{2+\sqrt{3}}\frac{\ln a}{1+a^2}da+\int_0^{2+\sqrt{3}}\frac{\ln a}{1+a^2}da$$

$$\Rightarrow I_2=\frac{\pi}{2}\ln 2+\frac{5\pi}{12}\ln\left(2+\sqrt{3}\right)+\frac{\pi}{2}\int_0^{2+\sqrt{3}}\frac{a}{1+a^2}da=$$

$$=\frac{\pi}{2}\ln 2+\frac{5\pi}{12}\ln\left(2+\sqrt{3}\right)+\frac{\pi}{4}\ln(1+a^2)\Big|_0^{2+\sqrt{3}}=$$

$$=\frac{\pi}{2}\ln 2+\frac{5\pi}{12}\ln\left(2+\sqrt{3}\right)+\frac{\pi}{4}\ln\left(4\left(2+\sqrt{3}\right)\right)$$

$$\Rightarrow I_1=\int_0^1\frac{(x^2+1)\ln(1+x)}{x^4-x^2+1}dx=\frac{\pi}{6}\ln\left(2+\sqrt{3}\right)$$

$$I_2=\int_0^1\frac{1+x^2}{1+x^6}\ln x\,dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{6n}(1+x^2)\ln x\,dx$$

$$\int_0^1 x^k\ln x\,dx=-\frac{1}{(k+1)^2}\Rightarrow I_2=\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{1}{(6n+1)^2}+\frac{1}{(6n+3)^2}\right)$$

$$=-\frac{1}{36}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{6}\right)^2}-\frac{1}{9}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-\frac{1}{144}\left(\psi_1\left(\frac{1}{12}\right)-\psi_1\left(\frac{7}{12}\right)\right)-\frac{G}{9}$$

$$9\sum_{k=0}^{\infty}\frac{1}{(3x+k)^2}=\sum_{k=0}^{\infty}\left(\frac{1}{\left(x+\frac{3k}{3}\right)^2}+\frac{1}{\left(x+\frac{3k+1}{3}\right)^2}+\frac{1}{\left(x+\frac{3k+2}{3}\right)^2}\right)$$

$$\Rightarrow 9\psi_1(3x)=\psi_1(x)+\psi_1\left(x+\frac{1}{3}\right)+\psi_1\left(x+\frac{2}{3}\right)$$

And using: $\psi_1\left(\frac{1}{4}\right)=\pi^2+8G;\ \psi_1\left(\frac{3}{4}\right)=\pi^2-8G$

$$9\psi_1\left(\frac{1}{4}\right)=\psi_1\left(\frac{1}{12}\right)+\psi_1\left(\frac{5}{12}\right)+\psi_1\left(\frac{3}{4}\right)\Rightarrow\psi_1\left(\frac{1}{12}\right)+\psi_1\left(\frac{5}{12}\right)=8\pi^2+80G$$

With the reflection formula: $\psi_1(x)+\psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$ ***we have:***

$$\psi_1\left(\frac{5}{12}\right)+\psi_1\left(\frac{7}{12}\right)=\frac{\pi^2}{\sin^2\left(\frac{5\pi}{12}\right)}\Rightarrow\psi_1\left(\frac{5}{12}\right)=4(2-\sqrt{3})\pi^2-\psi_1\left(\frac{7}{12}\right)$$

$$\Rightarrow\psi_1\left(\frac{1}{12}\right)+4(2-\sqrt{3})\pi^2-\psi_1\left(\frac{7}{12}\right)=8\pi^2+80G\Rightarrow$$

$$\Rightarrow\psi_1\left(\frac{1}{12}\right)-\psi_1\left(\frac{7}{12}\right)=4\sqrt{3}\pi^2+80G$$

$$I_2=-\frac{1}{144}\left(4\sqrt{3}\pi^2+80G\right)-\frac{G}{9}=-\frac{\pi^2}{12\sqrt{3}}-\frac{2}{3}G$$

$$\Rightarrow I_1-I_2=I=\frac{\pi}{6}\ln\left(2+\sqrt{3}\right)+\frac{\pi^2}{12\sqrt{3}}+\frac{2}{3}G$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru